THE

MATHEMATICAL GAZETTE.

EDITED BY
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WITH THE CO-OPERATION OF
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The Mathematical Association.

ANNUAL MEETING, 1913.

The Annual Meeting was held at the London Day Training College, Southampton Row, London, W.C., on 8th January, 1913, the President, PROFESSOR E. W. HOBSON, Sc.D., F.R.S., in the Chair.

MORNING MEETING.

PRESIDENTIAL ADDRESS.

ON GEOMETRICAL CONSTRUCTIONS BY MEANS OF THE COMPASS.

THE subject that I have chosen for my address is a special point connected with the theory of the constructions of Euclidean Geometry. I propose to shew, in a simple manner, that the essential elements of all the constructions made in this part of Geometry can be obtained by the employment of the compass alone, without the aid of the ruler. Before proceeding, however, to this main point of my address, I desire to make a few remarks upon the general theory of the constructions of Euclidean Geometry.

The usual statement made, when it is wished to assign the scope of the Euclidean constructions, is that the constructions are such as can be made by employing two instruments only, the ruler and the compass. This statement suffers from the serious defect that it takes no account of a certain fundamental distinction. The neglect of this distinction not only leads to confusion of thought in relation to the science of Geometry, but it has also been a fruitful source of misunderstandings as regards the real nature of some well known special problems of Geometry that have attracted the interest of a great number of men throughout many centuries. The distinction to which I allude is that between abstract or theoretical Geometry on the one hand, and practical or physical Geometry on the other hand. On the practical side, Geometry is a physical science in which the objects dealt with—points, straight lines, circles, etc.—are physical objects, the relations between which are to be ascertained and described, with a view to dealing with actual spatial relations with an accuracy sufficient for all practical purposes. On the theoretical side, Geometry is an abstract science concerned with the relations between objects, which, although they are called by the

same names—points, straight lines, circles, etc.—as before, are no longer physical objects. The properties and relations of these ideal objects are assigned by means of a scheme of definitions and postulations suggested by the observations made in physical Geometry. These objects are idealisations, obtained by abstraction, of those physical objects that bear the same names. The properties and relations in theoretical Geometry are absolutely precise, whereas the corresponding properties and relations are realised only in some greater or less degree in any physical objects which we may take as representative of the ideal objects. In fact, there is an inevitable element of inexactness in all our actual spatial perceptions which is eliminated when we pass from physical to abstract Geometry. The obliteration of the distinction between abstract and physical Geometry is furthered by the fact that we all of us, habitually and necessarily, consider both aspects of the subject at one and the same time.

We may be thinking out a chain of reasoning in abstract Geometry, but if we draw a figure, as we must do in order to fix our ideas, it is excusable if we do not always remember that we are not in reality reasoning about the objects in the figure, but about objects which are their idealisations and of which the objects in the figure are only an imperfect representation. Even if we only visualise, we see the images of physical objects which are only approximative representations of those objects about which we are reasoning. It would take me too far into the philosophy of the subject if I were to attempt to give the grounds upon which the necessity rests, of rising from a practical to a theoretical treatment of the subject of spatial relations, as an essential condition for a really fruitful scientific development of the subject of Geometry.

Rulers are physical objects by means of which physical straight lines, of approximate straightness and of small but uncertain thickness, can be constructed. A compass is a physical object by means of which physical circles, of small thickness, can be constructed. When we say then that Euclid's constructions are such as can be made with a ruler and a compass, it is clear that the statement is primarily one relating to the physical constructions which lead to the counterparts and the practical representations of the ideal objects with the determination of which Euclid, in his abstract Geometry, is concerned. The assertion is only a completely accurate one if it be taken to mean that the Euclidean postulates as to the existence of straight lines and circles imply that the corresponding practical constructions can be made with an indefinite degree of accuracy, subject only to the limitations of the instruments employed, the ruler and the compass. The unconscious assumption that the converse of this is true, namely that any construction that can be made practically by means of the ruler and compass, to an unlimited degree of approximation, necessarily corresponds to an ideal Euclidean construction or rather determination, allowable in accordance with the Euclidean postulates, is an error which, I think, accounts in a large measure for the aberrations of the circle squarer and of the trisector of angles. These two problems, that of squaring the circle and that of trisecting an angle, are soluble as problems of practical Geometry, by employing the ruler and compass, even with such accuracy that the errors would be imperceptible; but the corresponding ideal problems are, as we now know, incapable of solution by Euclidean modes of determination. Thus it is not true that any problem of construction which can in practical Geometry be solved by means which require only the use of the ruler and compass has, corresponding to it in abstract Geometry, a problem that can be solved by methods restricted in the Euclidean mode.

Let us consider what the Euclidean postulations in this matter amount to. In the first place it is postulated that any two assigned points A, B determine uniquely a straight line (A, B); the whole straight line, not merely the segment between A and B; the points A and B being incident on this

straight line. It is further postulated that a unique circle $A(B)$ exists, of which A is the centre, and on which B is incident; similarly there exists the circle $B(A)$. To say that these postulations of existence amount to allowing the use of the ruler and compass is to leave the abstract stand-point and to pass over to that of practical Geometry. In the abstract, as long as we are assured of the existence of the straight line (AB), or of the circle $A(B)$, as determinate objects of known nature, we have all we want; we can use them and reason about them and about their relations to other entities; the notion of drawing or constructing them is a notion appropriate to the physical, not the abstract, side of the subject. But we have not yet made clear what the effect of the complete Euclidean scheme of postulations is, in relation to problems of what is usually called construction, but would be better called determination. Every such problem is in its essence reducible to the determination of a certain number of points which shall satisfy certain conditions prescribed in the problem. How then are we to be allowed to consider a point as determined? When this question has been answered we then know the exact scope, and the limitations, of the possibility of Euclidean constructions. In Euclidean Geometry a point is determined in three ways only, in accordance with the following rules:

If A, B, C, D are four assigned points, then—

(1) There exists in general one point P, regarded as determinate, which is incident on both the straight lines (A, B) and (C, D), called their intersection. In case the four points are all incident on one straight line this determination fails, as also if (A, B) and (C, D) are parallels.

(2) A point P which is incident both on the straight line (A, B) and on the circle $C(D)$, when such a point exists, is regarded as determinate.

(3) A point P which is incident both on the circle $A(B)$ and on the circle $C(D)$, when such a point exists, is regarded as determinate.

Whenever, in a Euclidean problem of determination, a point is to be determined, it must be given by the use of the rules (1), (2), (3), these rules being repeatedly used any finite number of times. In each case in which one of these rules is employed it must be shewn that it does not fail to determine the point. In this last requisite Euclid's own treatment of problems is sometimes defective, as for example in the first proposition of the first book, where (3) is employed without proof that the two circles intersect one another.

The practical constructions corresponding to (1) involve the use of the ruler only, those corresponding to (3) require the use of the compass only, and those corresponding to (2) require the use of both ruler and compass.

It is an interesting question whether all three modes of determining a point are really necessary for the problems of construction, or whether any restrictions on their use may be made without diminishing the range of the problems that can be solved. In fact, more than one such restriction can be made. I propose to shew you in detail that all the Euclidean constructions or determinations can be made by means of (3) alone, without the use of (1) or (2); that is to say, I shall shew that any point that can be determined by (1) or by (2) can also be determined by (3). In other words, any point required for the purposes of a Euclidean construction can be determined as an intersection of two circles. In practical Geometry this means that the compass alone suffices for all the constructions that can be made by means of ruler and compass. All the essential points of a required figure may be obtained by using the compass alone; the ruler may be completely discarded, unless, after the cardinal points of the required figure have been obtained, we desire to fill in the straight lines of the figure, by joining pairs of the cardinal points, for which purpose we must naturally have recourse to the ruler.

In order to shew that the rules (1) and (2) are reducible to (3), it is necessary to solve a short chain of four problems by means of (3), as follows:

(α) Fig. (1) If P, A, B are three given points, it is required to find the image of P in the straight line (A, B). This image P' is given as the second point of intersection of the two circles $A(P)$, $B(P)$, and is thus, in accordance with (3), determinate.

(β) Fig. (2) If A, B are two given points, it is required to find by means of (3) a point B' on the straight line (A, B) such that $AB' = 2AB$, or such that AB' is a given integral multiple of AB.

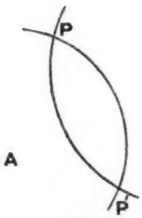

FIG. 1.

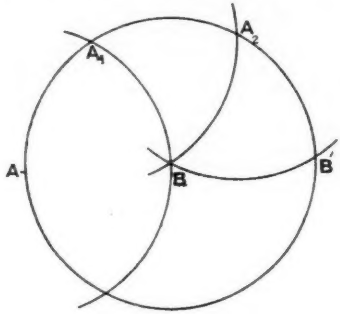

FIG. 2.

Determine A_1 as an intersection of $A(B)$ with $B(A)$, then determine A_2 as the intersection of $A_1(B)$ with $B(A)$. Lastly A', the other extremity of the diameter of the circle $B(A)$ through A, is determined as the intersection of $A_2(B)$ with $B(A)$. The point A' is the point required, such that $AA' = 2AB$. By continued repetition of this procedure it is easy to see that a point $A^{(n)}$ on the straight line (A, B) can be determined, so that $AA^{(n)} = n \, . \, AB$, where n is any assigned integer.

(γ) Fig. (3) Having given three points A, B, P, it is required to determine by means of (3) the image, or inverse point of P, with respect to the circle $A(B)$.

Determine Q and Q' as the intersections of $A(B)$ with $P(A)$. Determine P' as the second point of intersection of $Q(A)$ and $Q'(A)$.

The triangles $AP'Q$, AQP are similar, and thus $\dfrac{AP'}{AQ} = \dfrac{AQ}{AP}$, or $AP \, . \, AP' = AQ^2$, hence P' is the required inverse point. This method fails in case AP is less than half the radius of the circle. In that case we employ (β) to determine the point $P^{(n)}$ on AP, where $AP^{(n)} = n \, . \, AP$, n being so chosen that $AP^{(n)}$ is greater than half the radius of the circle. We then as before determine the inverse point $P'^{(n)}$ of $P^{(n)}$ with respect to the circle.* Lastly, we determine P' by means of (β), so that $AP' = n \, . \, AP'^{(n)}$.

(δ) Having three points A, B, C assigned, it is required to determine by means of (3), the centre of the circle on which they are incident.

Let C', Fig. (4), the inverse point of C with respect to the circle $A(B)$, be determined by (γ). Let D', the image of A in the straight line (B, C') be determined by (α). Let D, the inverse point of D' with respect to the circle $A(B)$, be determined by (γ); then D is the centre of the circle on which A, B, C are incident. This can be proved in a simple manner, if it be remembered that the inverse, with respect to $A(B)$, of the circle on which A, B, C are incident is the straight line BC'.

We are now in a position to shew that the methods (1) and (2), of deter-

* The construction here given may be employed to bisect the straight line (AB). Determine by means of (β) the point B' incident on (AB) such that $AB' = 2AB$. The point required can then be determined by (γ) as the inverse point of B' with respect to the circle $A(B)$.

mining a point, are reducible to the method (3). First, as regards (1), in which the point is to be determined by the condition that it shall be incident on each of the straight lines (A, B), (C, D).

Take any circle whose centre O is not incident on (A, B) or on (C, D). Determine, by means of (γ), the points A', B', C', D' which are the inverse points of A, B, C, D with respect to this circle. Determine, by means of (δ), the centre H of the circle on which O, A', B' are incident, and also the centre K of the circle on which O, C', D' are incident. Employing (3), the second point of intersection P' of the two circles $H(O)$, $K(O)$ is determinate. The required point P is the inverse point of P' with respect to the circle of centre O, and can therefore be determined by means of (γ). The proof is simply carried out by employing the known properties of inverse figures.

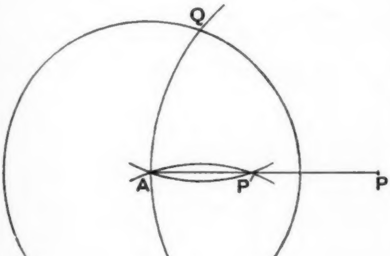

FIG. 3.

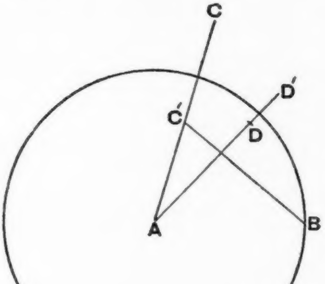

FIG. 4.

Next consider (2), in which the point is to be determined as one which is incident on the straight line (A, B) and on the circle $C(D)$. Determine, by means of (β), the points A', B' which are the inverse points of A, B with respect to the circle $C(D)$. Then, by means of (δ), determine the centre O of the circle on which C, A', B' are incident. Determine, by (3), the intersections, P', Q' of the circles $C(D)$ and $O(C)$. The required points P and Q, the intersections of (A, B) and $C(D)$ are now determined by means of (β) as the inverse points of P' and Q' with respect to the circle $C(D)$. This method fails in case the C is incident on (A, B). In that case we must have recourse to an inversion with respect to some other circle of which the centre is not on (A, B).

It has now been completely established that any required point that can be determined as the intersection of two straight lines, or as an intersection of a straight line and a circle, may be determined as an intersection of two circles. In practical Geometry, the use of the compass alone is therefore sufficient for the determination of the cardinal points of a required figure.

The first treatment of Euclidean constructions by means of the compass without the ruler was given by L. Mascheroni in a work entitled *Geometria del compasso* that was published in 1797 at Pavia. This work contains the solution of a large number of problems by the employment of the compass alone, and is interesting on account of the ingenuity displayed in many of the solutions. Various other treatises and memoirs on the same lines have been since published. Among these I may mention a paper by L. Gérard (*Math. Ann.*, vol. 48, 1897), in which a construction of the kind we have been

considering is given for the division of the circumference of a circle into 17 equal parts. The first writing, however, in which a systematic proof was given of the possibility of making all the Euclidean constructions by means of the compass alone is contained in a paper by A. Adler (*Zur Theorie der Mascheronischen Konstruktionen*, Wiener Sitzungsberichte, vol. 99, 1890). In this paper the method of inversion, which I have made the basis of the treatment in the present address, is systematically employed.

In the case of any particular problem of construction the figure would be of quite unwieldy complication if the determination of each of the required points were carried out by working through all the steps in accordance with the procedure I have indicated. This will, however, in any particular case be unnecessary; by the exercise of some geometrical dexterity a comparatively simple construction for attaining the required end will be discoverable. My object, in indicating the solutions of the short chain of problems by the method described, was to provide a proof of the possibility of solving all problems of Euclidean construction by means of the compass, but I did not intend to lay down a process which should be carried out in each particular problem of construction.

As regards other restrictions upon the unlimited use of the ruler and compass which may be introduced without diminution of the range of soluble problems, I may mention that, as was shewn by Geometers in the sixteenth century, all the Euclidean constructions can be made by means of a ruler and a compass with its arms making a fixed angle with one another. It was shewn by Brianchon and his followers that many constructions in which ruler and compass are usually employed can be made by the use of the ruler alone; more recently the limits of the possibilities in this direction have been carefully examined. The remarkable fact was established by Poncelet and Steiner that if we have given, once for all, a single fixed circle, with its centre known, then all the Euclidean constructions can be carried out by employing the ruler alone. Although not all the Euclidean constructions can be carried out by the use of the ruler with a single straight edge alone, it can be shewn that all these constructions can be carried out by employing a ruler with two parallel straight edges, or even by employing a ruler with two straight edges inclined to one another at any angle (not 180°).

Time does not allow me to prove the truth of these last statements, but I have indicated a field in which the working out of detailed problems affords unlimited scope for the exercise of the ingenuity of those persons, of which the number is much larger than is often supposed, who are endowed by nature with an interest in the special problems of elementary Geometry.

Even for purely didactical purposes, some use, I think, might be made of ideas of this order. Many of the recent changes in the methods and in the subject matter of mathematical teaching have been such as to leave less scope than formerly for the display, on the part of students, of any ingenuity they may possess. No doubt, ingenuity is not one of the faculties possessed in any considerable degree by the Intellectual Democracy. Our methods of teaching and the matter taught are very properly chosen chiefly with a view to meet the needs of the average student, and have regard, in the main, to average capabilities. There are, however, individual students in every institution who do rise above the average in respect of natural interest in, and capacity for, a particular study such as Geometry. For these, a strictly moderate amount of time spent on problems which afford scope for some ingenuity will be well spent, and will at least perform the important function of stimulating in them a real interest in matters which are apt otherwise to strike them as being so purely mechanical in their nature as to afford little or no scope for initiative.

Having read his address, on which there was no discussion, the **President** called upon **Mr. E. M. Langley** for his paper on "Map Projections."

MAP PROJECTIONS.

Mr. E. M. Langley, in introducing his subject, said: Before having the slides thrown on to the screen, I should like to say a few words as to how I came to have this set of slides for exhibition. At Bedford we have an Arts' Club, and I was asked by this club to give a paper, and on one particular occasion took map projections as the subject for my little discourse. It occurred to me that possibly these slides might be deposited in the Library here and lent to anybody who ever happened to want such a set. But it would be of very little use to say that such slides were handed over to the Library, because nobody would know what they were like, and I thought it would be better to show them to a meeting of the Association, so that some idea could be formed as to whether they would ever be of any possible use, and, with various corrections and additions, suit the purpose of some one taking up the same subject.

I thought I might also take the opportunity of mentioning various works and articles which I have found useful. I did not, when I offered the subject to the Council, know the recent issue of A. R. Hinks' book on *Map Projections*. I should like to say that as far as I can judge this is an excellent book, and that wherever you go in it you will find something that you want to see, and, moreover, you will find it explained in clear and lucid language which, I may say, is not the case with many other treatises that I have consulted, especially in some of the older works, where the matter is frequently all right, but it takes one a long time to see what the writer is aiming at.

The works I have found most useful are De Morgan's *Doctrine of the Sphere*; Proctor's article on *Equigraphic Projections of the Globe* (in the *Intellectual Observer* for 1866); Herschel's *Treatise on Astronomy*; and *The Construction of Maps and Globes* by some unknown writer, 1717.*

I have also used an article which appears in the *Encyclopaedia Britannica*, in the last issue but one. I have not scrupled to draw on some old works which are out of print. You will therefore possibly see some old friends, and you must excuse me for having taken advantage of the diagrams in the books, for I have thus saved myself the trouble of drawing. Considerations of copyright have withheld me from reproducing some from more modern books. I have, however, a slide from a diagram in Martin's book on *Astronomy and Navigation*, which Messrs. Longmans gave me permission to use if I undertook to say I did so by their permission.

Mr. Langley, having explained the diagrams as they appeared upon the screen,

Professor Alfred Lodge asked if sailors use any projection except the Mercator. Did they use the stereographic?

Mr. Langley: It could be used, but I am not sufficiently a practical navigator to know whether they could cut anything short by means of it. The great point with the Mercator is that you can lay a rule across it and get your course by that means. I believe they use great circle charts with adaptations to the Mercator; that is to say, points A, B, C, D, etc., of a great circle course are marked on the Mercator and the runs $AB, BC, CD, \dots$ are made along rhumb lines.

* In addition to the above, I should have mentioned Major C. F. Close's *Text-Book of Topographical and Geographical Surveying* (H.M. Stationery Office. 3s. 6d.); Prof. A. Lodge's article on *Orthographic Projection* and Prof. Heawood's article on *Stereographic Projection* (*Mathematical Gazette*, vol. i. pp. 98-101); and Mr. C. S. Jackson's MSS., kindly lent to me some years ago. For useful details in dealing practically with the perspective projections, some older works, still occasionally to be picked up at second-hand stalls, may be consulted with advantage, such as Martin's *System of Mathematical Institutions* (1764), Nicholson's *Popular Course of Mathematics* (1835), and Jamieson's *Treatise on the Construction of Maps* (1814).

AFTERNOON MEETING.

Professor E. W. Hobson, Sc.D., F.R.S. (President of the Association), in the Chair.

The meeting was resumed at 2.30, when the President read the Report of the Council for the year 1912.

REPORT OF THE COUNCIL FOR 1912.

During the year 1912, 86 new members have been elected; the number now on the Roll is 737. Of this number 8 are honorary members, 31 are life members by composition, 50 are life members under the old rule, and 648 are ordinary members. The number of associates remains at about 200.

The Council regret to have to record the deaths of six members during the year: the Honourable Lady Welby; Dr. H. T. Bovey, F.R.S., a vice-president of the Association, and formerly Director of the Imperial College of Science and Technology; the Rev. C. Elsee, of Rugby, who was one of the original members of the Association for the Improvement of Geometrical Teaching at its formation in 1871; Mr. S. A. Saunder, of Wellington College, at one time Honorary Secretary of the Royal Astronomical Society, and Gresham Professor of Astronomy; Mr. R. A. M'Cullough, of the Otago Boys' High School, Dunedin, New Zealand; Mr. E. W. Floyd, Headmaster of King Edward's Grammar School, Aston, Birmingham; and Mr. A. Cockshott, who was for 30 years a master at Eton, and had been a member of the Association since 1878.

In connection with the International Congress on the Teaching of Mathematics, which was held at Cambridge in August, 1912, this Association organised an exhibition of books, models, and school work. The exhibition formed an interesting feature of the Congress, and had many visitors. A copy of the catalogue of the exhibits has been sent to each member of the Association. The thanks of the Association are due to the exhibitors for their hearty co-operation in the attempt to produce something worthy of the occasion and of the many eminent mathematicians who assembled from all parts of the world. The Council desire to make special mention of the assistance which was received from Messrs. Teubner, of Leipzic, and Messrs. Gauthier-Villars, of Paris, who forwarded large and beautiful collections of German and French mathematical works; also from Professor Gray, Dr. J. G. Gray, Sir George Greenhill, Colonel R. L. Hippisley, Mr. G. Goodwill, Mr. E. M. Langley, Mr. J. P. Kirkman, M. Vuibert, Messrs. Burroughs, the Brunsviga Company, the Felt and Tarrant Manufacturing Company, Messrs. C. and E. Layton, and the National Cash Register Company, who gave demonstrations of their respective exhibits. The exhibit of work from the public

and other secondary schools was collected and arranged by Mr. F. C. Boon. Mr. F. J. W. Whipple organised the exhibit of calculating machines, which was very complete; Miss Punnett and Dr. T. P. Nunn that of models used in elementary teaching. Mr. P. Abbott not only secured an extensive display of mathematical books and models, but undertook the duties of honorary secretary to the Committee, of which Mr. C. S. Jackson was chairman. The success attained is due in large measure to the exertions of those whose names are mentioned above, and in particular to Mr. Abbott, who, by his energy and patience, overcame many difficulties which might easily have led to the failure of the undertaking.

In the July number of the *Mathematical Gazette*, which was issued at the time of the International Congress, portraits were given of the chief officers of the Congress: Sir George Darwin, F.R.S., Sir Joseph Larmor, M.P., F.R.S., Professor E. W. Hobson, F.R.S., and Professor A. E. H. Love, F.R.S. A photograph of the statue of Sir Isaac Newton served as a frontispiece to the number.

The publication of its hundredth number, which took place last October, is an interesting event in the history of the *Gazette*. In order to celebrate the occasion, a special number of the *Gazette* has been prepared under the editorship of Professor Alfred Lodge, with the co-operation of Mr. C. S. Jackson. It contains portraits of Mr. E. M. Langley, the originator of the *Gazette* and its editor for the first year, and of Mr. W. J. Greenstreet, who has been sole editor since February, 1899. Brief biographical notes are appended.

A catalogue of the books in the Library has been completed during the year, and a copy has been sent to each member of the Association.

The Committee on the Teaching of Mathematics was reorganised in March in accordance with the scheme approved by the Association at its last annual meeting. Mr. A. W. Siddons has been elected chairman of the General Committee, with Mr. P. Abbott (in succession to Dr. T. P. Nunn) as honorary secretary. Mr. A. Lodge and Mr. C. V. Durell were elected to the corresponding positions on the Public Schools Committee; Mr. C. J. L. Wagstaff and Mr. W. J. Dobbs on the 'Other Secondary Schools' Committee; and Miss E. R. Gwatkin and Miss M. Punnett on the Girls' Schools Committee. Members of the Association who have matters to lay before any committee are requested to communicate with the honorary secretary of that committee.

The Council consider it to be desirable that the honorary secretary of the General Committee should be an *ex officio* member of the Council, and they recommend that Rule VII. be altered accordingly. The number of unofficial members of the Council

would remain the same as before, but the total number of members would be increased by one.

The various branches of the Association have been active throughout the year, and have done useful work.

Mr. F. J. W. Whipple and Mr. P. Abbott now vacate their seats as ordinary members of the Council in compliance with the rules of the Association, and the members present at the annual meeting are asked to nominate and elect their successors. Mr. Abbott will become an *ex officio* member of the Council, if the alteration of the rules is approved.

Professor E. W. Hobson, Sc.D., F.R.S., retires at this meeting from the office of President, which he has held for two years. The Council, in the name of the Association, desire to record their sense of the debt which is owing to Professor Hobson for the vigorous assistance which he has given to the Association in every direction in the prosecution of its work. As Professor Hobson's successor, the Council have the honour to nominate Sir George Greenhill, F.R.S., to be President for the years 1913 and 1914. They desire also to nominate Professor Hobson to be an Honorary member and a Vice-President of the Association.

The Council have again to express their hearty appreciation of Mr. Greenstreet's work as editor of the *Mathematical Gazette*; and to thank the authorities of King's College and of the London Day Training College for their kindness in affording accommodation for the numerous meetings which have been held in connection with the Association's work.

Reports from the Branches at London, Bangor, Southampton and Sydney (N.S.W.), have appeared in the *Mathematical Gazette* from time to time. The following special reports have also been received:

THE LONDON BRANCH, 1912.

The London Branch has held six meetings during the year, all of which have been well attended. At the Annual Meeting in February, Professor M. J. M. Hill delivered an address on "The Theory of Proportion." At other meetings papers have been read on "The Mathematics Syllabus in German Schools," "The Introduction of Algebra into the Mathematics Course," and the "Correlation of the Teaching of Mathematics and Geography." Special attention has been paid to Arithmetic, as a consequence of two papers on "Commercial Arithmetic" and "The Arithmetic of Citizenship."

The branch has appointed a special committee to investigate and report upon the syllabuses for the teaching in schools of Commercial Arithmetic and the Arithmetic of Citizenship. At the November meeting Mr. C. S. Jackson gave a short account of the recent International Congress on the Teaching of Mathematics; and the branch is also indebted to him for an interesting paper showing how intimately connected Mathematics is with the sport of pigeon racing. The Annual Social Meeting was held at the Regent Street Polytechnic on Nov. 30th. Over 120 members and friends

were present, and a most enjoyable evening was spent. In addition to exhibits of text-books and apparatus, the programme included music and singing, cinematograph films, and a lantern lecture by Mr. W. M. Roberts on "Alpine Climbing."

THE SOUTHAMPTON BRANCH, 1912.

THE Southampton Branch have had an interesting and successful year under the Presidency of Professor E. L. Watkin, of the Hartley University College.

Five papers have been read as follows:

26 Jan. Modern Mathematical Methods, by Mr. W. T. Tregear, B.A.

23 Feb. The Principles of Mechanics, by Mr. W. D. Evans, M.A.

22 Mar. The Teaching of Mechanics, by Mr. J. B. Crompton, M.A.

18 Oct. The Mensuration of Solids, by Professor E. L. Watkin, M.A.

29 Nov. Geometry, a concrete application on the Theory of Order, as defined by boundaries, by Major E. T. Dixon, M.A.

Three of these papers were of a practical character, and calculated to assist members in their own class work. The papers of 23rd Feb. and of 29th Nov. were of a highly theoretical character, and proved a severe tax on the attention and comprehension of the hearers.

The meetings are always of a somewhat informal character. When ladies are absent the members indulge in smoking. The discussions are always keen and animated, though they very often turn upon side issues. The Branch is doing useful work, and at the least it gives those engaged in mathematical work in Southampton the opportunity of knowing one another, and of comparing ideas and methods of teaching.

The Treasurer stated that he had distributed copies of the Balance-sheet amongst those present, and would be glad to answer any questions that might be put.

The Report and Balance-sheet were adopted.

The President called attention to the words in the Council's Report: "The Council consider it to be desirable that the Honorary Secretary of the General Committee should be an *ex-officio* member of the Council, and they recommend that Rule VII. be altered accordingly."

This recommendation required the approval of the meeting.

Its adoption was proposed by Mr. Pendlebury, duly seconded, and carried unanimously.

Sir George Greenhill, F.R.S., was elected President for the years 1913 and 1914.

Mr. Pendlebury proposed, Miss Punnett seconded, and it was carried unanimously: "That Professor E. W. Hobson, Sc.D., F.R.S., be elected an Honorary Member and a Vice-President of the Association."

Mr. W. E. Paterson (Mercer's School), and Mr. C. O. Tuckey (Charterhouse) were elected to the seats on the Council vacated by Mr. F. W. Whipple and Mr. P. Abbott, Mr. Abbott to become an *ex-officio* member of the Council as Honorary Secretary of the General Teaching Committee.

GEOGRAPHICAL DISTRIBUTION OF MEMBERS.

A LIST of members of the Association on January 1st, 1913, classified according to locality, shows a total of 731, of whom England contains 556, Wales 29, Scotland 27, Ireland 15, Isle of Man 1, Channel Islands 1, South Africa 26, West Africa 2, Canada 8, Australia 9, New Zealand 4, West

Indies 1, Malta 1, India 17, Straits Settlements 1, Brazil 1, Egypt 4, Italy 1, Japan 3, Roumania 1, Russia 1, Syria 1, U.S.A. 21.

The 556 members in England are distributed thus: London 147, Beds 8, Berks 13, Bucks 11, Cambs 28, Cheshire 6, Cornwall 2, Cumberland 5, Derbyshire 5, Devonshire 13, Dorset 1, Durham 6, Essex 13, Glos 28, Hants 15, Herts 12, Hereford 2, Isle of Wight 3, Kent 23, Lancs 31, Leicestershire 4, Lincs 2, Middlesex 7, Monmouth 1, Norfolk 8, Northants 3, Northumberland 8, Notts 4, Oxfordshire 13, Rutland 4, Shropshire 7, Somerset 8, Staffs 5, Suffolk 1, Surrey 25, Sussex 17, Warwickshire 16, Westmoreland 1, Wilts 3, Worcs 8, Yorkshire 39. H. D. Ellis.

The **President** called upon **Mr. G. St. L. Carson** to read the following paper on "Intuition."

INTUITION.

If there be one duty more incumbent than any other upon mathematicians, it is to have a clear and common understanding of every term which they use. I do not say a formal definition, though that is most advisable if and when it can be obtained; but a class of entities must be known and recognised before it can be defined, and no term should be used unless it at least gives rise to definite, recognisable and identical images in the minds of the speaker and listener. It cannot fairly be said that mathematicians are at fault in this respect, when dealing with their own special subjects; but I fear they cannot so easily be acquitted when discussing the didactic side of their work. Concrete and utilitarian, axiom and postulate, intuition and assumption; how many of us have definite meanings for these terms, and can feel certain that they represent the same meanings to others? The term which I have chosen as the title of this paper is one of the most commonly used and, as it seems to me, most often misunderstood; at the same time, the ideas and processes for which it stands lie at the root of all elementary teaching. I have therefore thought it worth while to discuss its meaning, and to show the bearing of the process on mathematical education.

There is, I think, little doubt that to most of those who use the term intuition, it connotes some peculiar quality of material certainty. Take, for example, the equality of all right angles, or the angle properties of parallel lines, and ask one who understands these statements with what degree of certainty he asserts their truth. It will be found almost invariably that he regards them as far more certain than statements such as "the sun will rise to-morrow morning" or "all men are mortal"; these, he admits, might be upset by some perversion of the order which he has regarded as customary, but the geometrical statements appear to be of the essential nature of things, eternal and invariable verities. So much, indeed, is this the case that the very idea of practical tests is grotesque; who has ever experimented to ascertain whether, if two pieces of paper are folded, and the folds doubled again on themselves, the corners so formed are superposable? If the individual under examination be questioned as to the basis for this faith, he can only reply that it is the nature of things, or that he knows it intuitively; of the degree of his faith there is no doubt. It is to statements asserted in this manner that the term intuition is commonly applied; other facts, such as the mortality of all men, which are justified by the fact that all human experience points to them, are not classified under this heading nor, as I have said, are they accepted with the same faith.

These alleged certainties can of course be dissipated by purely philosophical considerations concerning the relations and differences between concepts and percepts; but "an ounce of practice is worth a ton of theory," and I propose here to show, mainly from historical considerations, that there is no ground for absolute faith in certain intuitions, however tenaciously they may be held. Take first the idea, still held by many, that a body in motion must be urged on by some external agent if its velocity is to be maintained. Until the time of Galileo this belief was held universally, even men of eminence who had considered the subject being convinced of its truth. Now this faith was of just such a kind, and just as strongly held, as the faith in geometrical statements which I have mentioned; it was (and is by many) regarded as in the nature of things that a body should stop moving unless it is propelled by some external agency. And yet others, of whom Galileo was the forerunner, see the nature of things in a light wholly different. They regard it as utterly certain that a body can of itself neither increase nor retard its own motion. Ask a clever boy who has learnt some mechanics, or even a graduate who has not thought over much on the foundations of the subject, which he regards as more unlikely; that an isolated body should, contrary to Newton's first law, set itself in motion, or that the secret of immortality should be discovered. He will tell you that the second might happen, though personally he does not believe that it ever will; but that a body can *never* begin to move unless it has some other body "to lever against." We thus see two contradictory intuitions in existence, each held with equal strength.

Coming to more recent history, let me remind you of the development of the theory of parallels, and the rise of non-Euclidean Geometries. Until the last century, it may fairly be said that no one had ventured to doubt the so-called truth of the parallel postulate, though many eminent mathematicians had endeavoured to deduce it from the other postulates of geometry. The genius of Bolyai and Lobachewsky, however, put the matter in quite another light. They showed that a completely different theory of parallels was just as much in accord with the nature of things as that hitherto held; and that, to beings with more extended experience or finer perceptions than ours, this different theory might appear to correspond with observation while that current failed to do so. In other words, they showed that there are several ways of accounting for such space-observations as we can make with our restricted opportunities; just as it was then well known that there were two theories which fitted the observations of astronomers, of which Newton's was the more simple and self-consistent.

It thus becomes clear that intuitions are no more than working hypotheses or assumptions; they are on the same footing as the primary assumptions concerning gravitation, electrostatics or any other branch of knowledge based on sensation. They differ from these in that they are formed unconsciously, as a result of universal experience rather than conscious experiment; and they are so formed in regard to those experiences—space and motion—which are forced on all of us in virtue of our existence. It is not implied that their possessor is even fully conscious of them; ask some comparatively untrained adult how to test rulers for straightness, and he may be at a loss or give some ineffective reply; but suggest placing them back to back and then reversing one, and he at once assents. He regards this not as new information, but as something so simple and obvious that it had not occurred to him. It is to him the essential nature of things; he has held this view from so early an age, and it has remained so entirely free from challenge, that he revolts at the suggestion that things, viewed from another standpoint, may appear to have a different nature.

The formation of such working hypotheses is the normal method by which the mind investigates natural phenomena. After observation of a certain set of events, a theory is formed to fit them, the simplest being chosen if more than one be found to fit the facts equally well. This theory is developed, and its consequences compared with the results of further observations; so long as these are in accord, and so long as no simpler theory is found to accommodate the fact, the first theory holds the field. But, should either of these events occur, it is abandoned ruthlessly in favour of some better description of the recorded observations. There are famous historical cases of each event; Newton's corpuscular theory of light yielded deductions in actual disaccord with observation, and was therefore abandoned. The ancient theory of astronomy, wherein the stars were imagined to be fixed on a crystal sphere and which the planets travelled in epicycles, was abandoned in favour of the modern theory, not because it could not be modified to accord with observation, but because of its greater complexity. In every such case the question of absolute truth is irrelevant and beyond our reach; the problem is to find the simplest theory in accord with all the facts, abandoning in the quest each theory as a successor is found which better fulfils these requirements.

Shortly, then, we may say that intuitions are merely a particular class of assumptions or postulates, such as form the basis of every science. They are distinguished from other postulates first, in that they, with their subject-matter—*e.g.* space or motion—are common from an early age to every human being endowed with the ordinary senses; and secondly, in that no other assumptions fitting the sensations concerned ever occur to those who make them. Their formation is forgotten, and they are therefore regarded as eternal; they hold the field unchallenged, and are therefore regarded as inviolable.

Before passing to the consideration of the bearing of intuition on the teaching of mathematics, it may be well to illustrate what has been said by the consideration of a few particular cases.

First, suppose that one sees a jar on a shelf, and puts his hand up to find out whether it is empty. Is the act based on an intuition from the appearance of the jar? This is not the case; if pressed before the act, one would not express any final certainty that the hand could enter the jar; it might have a lid or be a dummy. The individual can make more than one assumption which corresponds to the sight-sensation; the first chosen—that the hand can enter the jar—is merely the most likely as judged by experience.

Next suppose that a knock is heard in a room. The natural exclamation "what is that?" is based on intuition, for it expresses the now universal conviction that such a noise is an invariable accompaniment of some happening which, given opportunity, will also appeal to the other senses. Accumulation of human experience has led to the belief that such is invariably the case; but belief it is, and not certainty. If the reply were "it is nothing; under no circumstances could you have correlated any sight or feeling with that sound," it would be received with complete incredulity.

Consider again the statement that, given a sufficient number of weights, no matter how small, one can with them balance a single weight, however large. No one would doubt this or treat it as anything but the most obvious of truisms, and yet it is a pure assumption, formed unconsciously as the result of general experience; it answers in every respect to our definition of an intuition. It may be thought by some that the statement can be proved arithmetically, but in every such alleged proof the assumption itself will be found somewhere concealed.

We have, in fact, no warrant for assuming that the phenomenon called weight retains the same character, or even exists, for portions of matter which are so small as to be beyond our powers of subdivision.

Finally, consider the statement "I knew intuitively that you would come to-day." In what respect do those who use it regard it as differing from "I thought it was almost certain that you would come to-day"? It may fairly be said that the former expresses less basis of knowledge but more feeling of certainty than the latter; it means, "I don't in the least know how I knew it, but I did know beyond all doubt that you would come." Such ideas, with or without the use of the actual term intuition, are common enough. They are here quoted to justify the statement, made above, that the term connotes to many of those who use it some peculiar degree of certainty. Such statements are not intuitions; they are mere superstitions, and those who are subject to them fail to realise how often they are unjustified by the event. Belief in the absolute truth of the angle properties of parallels or of the Laws of Motion is equally a superstition; though these are, until now, justified by the event. The truth is that they can never receive this absolute justification, for no material observation is beyond the possibility of error, nor can it be certain that some simpler theory will not be formed, accounting equally well for the observations; it is the belief in this impossible finality which constitutes the superstition.

Turning now to the more educational aspect of the subject, the first problem which confronts us is this: children, when they commence mathematics, have formed many intuitions concerning space and motion; are they to be adopted and used as postulates without question, to be tacitly ignored, or to be attacked? Hitherto teaching methods have tended to ignore or attack such intuitions; instances of their adoption are almost non-existent. This statement may cause surprise, but I propose to justify it by classifying methods which have been used under one or other of the two first heads, and I shall urge that complete adoption is the only method proper to a first course in mathematics.

Consider first the treatment of formal geometry, either that of Euclid or of almost any of his modern rivals; in every case intuition is ignored to a greater or less extent. Euclid, of set purpose, pushes this policy to an extreme; but all his competitors have adopted it in some degree at least. Deductions of certain statements still persist, although they at once command acceptance when expressed in non-technical form. For example, it is still shown in elementary text-books that every chord of a circle perpendicular to a diameter is bisected by that diameter. Draw a circle on a wall, then draw the horizontal diameter, mark a point on it, and ask anyone you please whether he will get to the circle more quickly by going straight up or straight down from this point. Is there any doubt as to the answer?* And are not those who deduce the proposition just quoted, from statements no more acceptable, ignoring the intuition which is exposed in the immediate answer to the question? All that we do in using such methods is to make a chary use of intuition in order to reduce the detailed reasoning of Euclid's scheme; our attitude is that statements which are accepted intuitively should nevertheless be deduced from others of the same class, unless the proofs are too involved for the juvenile mind. We oscillate to and fro between the Scylla of acceptance and the Charybdis of proof, according as the one is more revolting to ourselves or the other to our pupils.

* There is often apparent doubt; but it will usually be found that this is due to an attempt to estimate the want of truth of the circle as drawn.

At this point I wish to suggest that a distinction should be drawn between the terms deduction and proof. There is no doubt that proof implies access of material conviction, while deduction implies a purely logical process in which premisses and conclusion may be possible or impossible of acceptance. A proof is thus a particular kind of deduction, wherein the premisses are acceptable (intuitions, for example), and the conclusion is not acceptable until the proof carries conviction, in virtue of the premisses on which it is based. For example, Euclid *deduces* the already acceptable statement that any two sides of a triangle are together greater than the third side from the premiss (*inter alia*) that all right angles are equal to one another; but he *proves* that triangles on the same base and between the same parallels are equal in area, starting from acceptable premisses concerning congruent figures and converging lines. The distinction has didactic importance, because pupils can appreciate and obtain proofs long before they can understand the value of deductions; and it has scientific importance, because the functions of proof and deduction are entirely different. Proofs are used in the erection of the superstructure of a science, deductions in an analysis of its foundations, undertaken in order to ascertain the number and nature of independent assumptions involved therein. If two intuitions or assumptions, A and B, have been adopted, and if we find that B can be deduced from A, and A from B, then only one assumption is involved, and we have so much the more faith in the bases of the science. Herein lies the value of deducing one accepted statement from another; the element of doubt involved in each acceptance is thereby reduced.

Next, to justify the statement that intuition has been attacked. Both Euclid and his modern rivals knew well enough that their schemes must be based on some set of assumptions; they differed only in the choice. Each agrees that intuitive assumptions are undesirable, but the modern school regard the extreme logic entailed by Euclid's principle of the minimum of assumption as impossible for young pupils. There is, however, a third school which pursues a different course; it professes to replace intuition by experimental demonstration. Pupils are directed to draw pairs of intersecting lines, measure the vertically opposite angles, and state what they observe; to perform similar processes for isosceles triangles, parallel lines, and so on. Instead of being asked "do you think that, if these lines were really straight, and you cut out the shaded pieces, the corners would fit?" they are told to find out, by a clumsy method, a belief which they had previously held, though it had never, perhaps, entered definitely into their consciousness. The question suggested is, in these homely terms, just sufficient to bring the idea before them, and it is at once recognised as according with the child's previous notions; he does not regard it as new, but merely as something of which he had not before thought so definitely.

It is this type of exercise in drawing and measurement which I regard as an attack upon intuition. It replaces this natural and inevitable process by hasty generalisation from experiments of the crudest type. Some advocates of these exercises defend them on the ground that they lead to the formation of intuitions, and that the pupils were not previously cognisant of the facts involved. But in the first place, a conscious induction from deliberate experiments is not an intuition; it lacks each of the special elements connoted by the term. And as to the alleged ignorance of the elementary idea of space, it appears to me to be a mistaken impression, based on undoubted ignorance of mathematical terminology. If you say to a child of twelve "Are these angles equal?" he has to stop to think first, what an angle is, and next, when angles are equal; by the time he has done this his mind is incapable of grasping

the peculiar relations of the angles in question, and he is labelled as ignorant of the answer. The real difficulty, and it is not a small one, is to lead the child to express familiar facts in precise mathematical terminology; to say "angles equal" rather than "corners fit." Until this terminology is thoroughly familiar, the effort of using it must absorb a large part of the child's attention, leaving little available for the matter in hand. This paper is not concerned with the methods or practice of teaching, but I would strongly urge all those who are concerned with young children to guard against this danger, by constant transition to and fro between common and technical phraseology, appealing at once to the former at the least sign of doubt or hesitation.* The learning of technical terms should not appear as part of the definite work, or it will inevitably be regarded as the major part; it should come incidentally and by gradual transition, as I have suggested.

The only alternative to this evasion or suppression of intuition is to accept it from the commencement as the natural basis for primary education. But to be of any avail, the acceptance must be unquestioned and complete; every intuition which can be formed by the pupils must without suggestion of doubt be adopted as a postulate, none being deduced from others, which are themselves no more easy of acceptance. Such a course leads, it need hardly be said, to considerable simplification in the early treatment of any subject. For example, in geometry the angle properties of parallel lines, properties of figures evident from symmetry, and the theory of similar figures (excluding areas) appear as postulates; in the Calculus it is not proved that the differential coefficient of the sum of a finite number of functions is equal to the sum of their differential coefficients; the statement is illustrated by, say, consideration of some expanding rods placed end to end, and at once commands acceptance. Here the question of terminology again arises; I have often been struck, in teaching schoolboys and students, by their slowness to accept this and similar results in the Calculus; the clue was given to me by a boy who remarked that it was taking him all his time to remember what a rate of increase was, and he could not manage any more at the moment. Since that time I have avoided many seeming difficulties with elementary and advanced pupils by appeal from technical to familiar terms, always of course rephrasing the result in the proper form before leaving the matter in hand.

It will, I know, be thought by many that this adoption of all natural intuitions involves an appalling lack of rigour. But I would ask those who are of this opinion to do one thing before passing judgment, and that is, to define and exemplify with some care the meaning of the term rigour. When they have done this, I think they may be disposed to agree with the answer to their accusation which I am now going to put forward. It is that the scheme suggested is perfectly rigorous, provided that every deduction made from the postulates adopted is logically sound; on the other hand, it is admitted that the mathematical training thus imparted is not complete, because no attempt has been made to analyse these intuitive postulates into their component parts, showing how many must perforce be adopted in the most complete system of deduction. In other words, we may be rigorous in regard to logical reasoning, or in regard to lessening the number of assumptions which form the basis of a science. The view for which I contend is, that in all stages of mathematical education, deductions from the assumptions

* It is no good to say, "Come now, what is an angle?" Appeal first to the tangible fact in the child's mind by saying, "Cannot you see that those corners *must* fit?" and then remind him that "equal angles" merely means the same thing.

made should be rigorous; but that in the earlier stages every acceptable statement or intuition should be taken as an assumption, the analysis of these, to show on how small an amount of assumption the science can be based, being deferred.

To avert misapprehension, let me say again that I propose that, when all intuitions are accepted as postulates, this should be done without question or discussion other than that necessary to give them some precision. To embark on a discussion of their nature, or to appear to cast doubt upon them, would be fatal; as fatal as has been the apparently futile process of deducing one accepted statement from another. The pupil is already in possession of a body of accepted truth; let us build on that and defer its analysis, or anything that pertains thereto, until he is sufficiently mature to appreciate the motive.

The first course of mathematics would, then, range from arithmetic and analysis through geometry to mechanics. In this last subject there is little scope for intuition. Most of the mechanical intuitions formed by the race as a whole have been mistaken, and it is just this fact which gives some indication of the proper commencement for the second course, in which the intuitive postulates are to be analysed and reduced as far as possible. Let the student learn something of the history of mechanics, realising that ideas which he regards as impossible and absurd were held, by men of great eminence, with faith just as strong as that which he places in his geometrical postulates. Then let it be suggested to him that this renders care in regard to assumption of vital importance, and so commence an analysis of the mechanical postulates, hitherto redundant, obtaining deductions of one from another to show their interconnection. This completed, and the task is not a large one, it is natural to suggest that the postulates of geometry deserve some examination, and so, according to the time available and the ability of the pupil, we may pass backward through a review of the foundations of geometry to an examination of the foundations of analysis and arithmetic. It is not, of course, implied that every student of mathematics can reach this goal; few can ever get beyond some consideration of the foundations of geometry, with a clear understanding of the end to be attained in its general application to all sciences. But I do wish to put forward, with such emphasis as I can, this general scheme of mathematical education; namely, an upward progress, based on intuition, from Arithmetic through Geometry to Mechanics, followed by consideration in the reverse order of the foundations of each branch, the upward progress constituting the first course, and the downward review the second course. It would, I believe, give an intelligible unity to the whole subject, and would do something to restore that purely intellectual appreciation which has so largely declined during the past generation.

Mathematics is a useful tool, but it is also something far greater, for it presents in unsullied outline that model after which all scientific thought must be cast. I have endeavoured to show how this outline may be developed, starting from those intuitions which are common to us all, and ending in an analysis demonstrating their true nature. The concrete illustrations, so necessary and illuminating in elementary teaching, are so many draperies, fashioned to render this outline visible to those who cannot otherwise appreciate it. Even the several branches, analysis, geometry, mechanics, serve the same end; behind them all is the one pure structure of mathematical thought. They who most appreciate the structure will best fashion the draperies, and so render it most clearly visible to those whom they instruct.

DISCUSSION ON MR. CARSON'S PAPER.

The President: The audience, I am sure, will be very grateful to Mr. Carson. I think it is within the experience of most of us that there are certain words about the meaning of which individuals are not quite certain. I mean that these words change their meaning from time to time, and one is not quite sure that other people read exactly the same meaning into them that one does oneself. As far as I am concerned, I may say that one of these words is the word "intuition." I have vaguely an impression that it has had a good many meanings at different times, and I do not feel quite sure that other people mean exactly the same thing as I do when I use it. I have certainly a strong impression that the commonest use of it has undergone almost a revolutionary change in the English language.

I believe that "intuition," as it was used by writers like Locke—intuitive ideas—meant almost the opposite from what it is largely taken to mean now. Locke's intuitive ideas were innate, inborn ideas; some property that the mind had itself, apart from experience. The idea of intuition was in that sense. But now it has come to have a meaning equivalent to the German word "Anschauung"; that is, something that you see—that is visible to you on the outside. That is exactly opposite to the meaning I spoke of as being used by Locke. Well, I was not absolutely sure that Mr. Carson was using it exactly in the sense I indicated, because he spoke of the certainty or uncertainty of intuition. If I have an intuition, it is quite certain it is something I see as it stands, and it is equally certain, unless I am lying, that if I say I see a thing, I do see it. There is no question of uncertainty about it. The uncertainty begins when you commence to interpret and generalise the intuitive.

Intuition is no use for scientific purposes until it is generalised; that is its first defect. We have got to put it into a scheme; and there is the difficulty, especially in regard to spatial intuition. As soon as we begin to generalise and to erect the result of what we see into general principles, the element of uncertainty comes in. It was so as regards the axiom of parallels. That was supposed to be derived from intuition. As a matter of fact, that was a false generalisation of intuition. We see lines over and over again that do not meet for a long way off, but the false generalisation came in when we asserted the existence in infinite space of lines that never meet, thereby inferring something that nobody ever did or could intuite.

The second defect of intuition is that it is inexact; it lacks precision. Before we can make use for scientific purposes of intuition, which I regard as separate acts of observation, we have to remedy these two defects. We have to draw general inferences from intuitional data, and to give them an exactitude which is not originally there; and in making the result of our intuitions exact, there is a considerable latitude as to the way in which we shall do it. And when we begin to generalise there is the possibility of error, so that I do not think the uncertainties can be said to reside in the intuitions themselves—in the separate acts—but in the process of generalising and erecting into a principle; in which case we very often assert that something is true which is of a far more general character than what was seen in the actual intuitions themselves.

I think that is one source of the difficulties in regard to intuition; but as regards teaching, the real question is this, "Where are we going to place those elements that come from intuition?" Euclid, and all the people who teach young children, allow elements to come in which are derived from intuition, and to come in sporadically at various points. Euclid certainly does that. I do not think he thought he did, but he does. (Laughter.) All the modern people who have written books on

geometry put the results of intuition at the beginning, instead of scattering them through the whole process. Then they call them postulations or axioms.

The question at issue between two schools is rather where these things are to be placed. There is no question that all science must be based on generalisations which were derived originally from intuition; but are all the statements which are the results of those generalisations to be placed at the beginning of the subject and followed by merely syllogistic treatment, that is, are we to make it a purely deductive matter? Or, on the other hand, are we to take the results of what we get from intuition as we want them throughout the different points in the subject?

Of course, for purely scientific purposes—for specialists—there are great advantages in the former of these methods; but I agree with what I understand is Mr. Carson's view—a view which I think would be held by almost everybody who has thought about what is possible in teaching young people; that it would be hopeless to put all those elements which are derived from intuition at the beginning and make the rest purely deductive, because one could never get young people to see why it was necessary to have so many postulations about order, and continuity, and various other things placed at the beginning. They would not see the object of them, and the result of any kind of teaching based on them would, I think, be a hopeless failure, except for people who are going to be more or less specialists. So that I think almost all teachers will agree that the intuition must come in at various points in the subject.

I think these are the only observations I have personally to make. I will now ask for more remarks on the very interesting paper we have heard.

Rev. E. M. Radford: Mr. President, you have just said that you do not understand, in connection with Mr. Carson's paper, in what sense he used the word "intuition." There is another word he used in regard to which I am in the same difficulty, and that is the word "circle." (Laughter.) What *is* a circle, and what does Mr. Carson understand by a circle? Anything intuitive or anything deductive, must, we know, be based on some sort of data or hypothesis, and the question is, "What is the idea of a circle in a child's mind?" If you ask a child in an elementary school, he will generally tell you that it is something round; that is to say he does not think of it primarily in connection with what we call the centre. Then, again, it is not quite certain whether he is thinking of the area or whether he is thinking of the perimeter; but I am certain that he thinks of the circle as being what we should usually speak of in mathematics as a curve of constant curvature rather than as a locus. And it is from that point of view that he would make this intuitive conclusion to which Mr. Carson points. If you regard a circle as the locus of a point which moves so that it is always at the same distance from a fixed point, the property mentioned by Mr. Carson is by no means an intuition. It requires proof; and Euclid very properly proves it from that point of view, because he starts with the definition of a circle. The whole thing, therefore, depends on what idea you have originally as to the particular geometrical figure which you are considering.

We always warn students in training colleges of the danger of definition. Of course the ordinary student generally tries to begin his lesson by giving a definition of what he is going to talk about, whereas the definition ought to come at the end, because it should include all the distinguishing features and all the fundamental ideas about this particular thing. And so one cannot very well begin with a definition of a circle. One must find out what ideas a boy has about a circle, first of all. One must get out of him the whole aggregate of his ideas about

circularity from every point of view; and when one has discovered that, proceed to frame a formal definition of a circle. This intuitive result would have occurred at a previous stage; and, therefore, when we get our definition, we should not go back to prove it. If, however, we *do* start from a formal definition, and make a formal set of hypotheses, then there are certain things which must be proved, which would be obvious if, at the outset, we regarded in a different way the particular figure about which we are talking. That is really the difficulty I feel with regard to intuition.

Of course it is easy enough to make anything obvious if you start talking about it from your own point of view. (Laughter.) Take, for instance, the definition of a conic section. If we are writing a book on geometry, we start by defining a conic in such a way as will enable us to get out of all difficulties we wish to avoid; and so, if we take half-a-dozen books, we shall find that there are half-a-dozen different definitions of a conic section. Certain definitions make certain properties obvious. We know that in the end all these definitions can be brought into agreement with one another; but we must define our initial position with regard to these things. That is my point. From some points of view certain results may be intuitive, but from other points of view they certainly are not.

Mr. G. F. Daniell: The subject is a very difficult one, and I think the importance of it was brought out very plainly in the Chairman's concluding remarks. The whole treatment depends on what intuitions we allow. The reference to a child having a perception of circularity reminds me that there is a whole field regarding this subject of intuitions which has been quite neglected by all speakers. The child, perhaps, has the advantage of most of us here, who are all mathematicians and, in that respect, somewhat disqualified for treating of intuitions, because it is the defect of our mathematical education that we have been the slaves of paper. We are too much visualists; we are too eye-minded; whereas the child is to a large extent muscle-minded, and a large number of important intuitions are connected with the muscular and tactile senses. That is the point that I wish to emphasise. The appreciation of constancy of curvature may be one of such intuitions.

Mr. Carson referred to mechanics as being a subject in which intuitions were found to break down—to lead us astray. Perhaps it is rather the mathematicians who are led astray. Is it not possible that the men who were actually engaged in piling up the stones and the rocks to build the Pyramids had a more correct set of intuitions of a mechanical sort than the geometers who made the plans? They had to employ their muscles, and they may have thus acquired fairly correct ideas upon the laws of motion.

At any rate I suggest that there are grounds for considering, especially with regard to the teaching of mechanics, those intuitions which are based on the muscular and tactile senses. I should like to consider, for example, how a child's notions of force and of time are developed, as I think in this way we should get some light to assist us in forming a psychological basis for the teaching of mechanics. (Applause.)

Mr. C. O. Tuckey: There have been several speakers who have agreed upon one thing—that intuitions differ very much in different persons. It has been suggested that we should be children, in order to get better intuitions from our muscular senses. I should like to ask Mr. Carson how he can base his teaching on intuitions if he is not sure whether the intuitions are common to his pupils. Surely these intuitions must differ in different persons, and therefore we must take a much smaller number of intuitions than will appeal to some of our pupils, because, in the case of other students, the number of intuitions as to which they feel confident will not be so large; that is to say, the differ-

ences between pupils must diminish the number of intuitions on which we are entitled to build. I should like Mr. Carson to give us a clear indication as to where he can draw the line in geometry, because I have seen it stated that Newton was discouraged from the study of Euclid for the reason that all the results there proved were intuitively obvious to him.

I suppose there must be great differences between one pupil and another, and it seems to me that unless we realize this, those who in intuition are least like Newton will suffer most.

Mr. Katz : In the examples of intuition that Mr. Carson gave us he was thinking almost exclusively of geometry. He did not refer to any other branch of mathematics. Perhaps the restriction of his examples to geometry arises from the fact that geometry belongs to two categories. It is, on the one hand, a science of spatial relations having our intuitions of space as its foundation, and, on the other hand, it belongs to pure mathematics as such, and might be regarded (as pure mathematics is regarded by philosophers of the subject) as a kind of symbolic logic.

Take the case of algebra and geometry; in pure algebra as such, you do not require any intuitions, because it consists of a chain of purely logical deductions. Of course, it is quite true that there are certain intuitions in arithmetic and also in algebra from which historically they have developed, but the point is that, as far as these sciences themselves are concerned, they are independent of the particular intuitions from which they arose.

I was struck with the point that in mechanics we have the intuition that a body will not continue in its motion in a straight line unless some force is applied to it. Mr. Carson pointed out that that was a view very commonly held, although it was unsound. I should differ from him. It is surely a very correct intuition. Bodies as we meet them in our ordinary experience are not the ideal bodies of mechanics at all. They are not isolated bodies, and such bodies, on account of the presence of friction, will continually require force in order to keep them in their motion. It seems to me that in geometry and mechanics the student may use the intuitions he possesses, so long as he bears in mind that these intuitions, and the systems of geometry or mechanics that are built upon them, are by no means coincident.

Mr. A. W. Siddons : It seems to me that the real difficulty is what to do with intuitions which one knows are not true. I find that boys frequently see things intuitively which I know to be untrue—hasty generalizations, in your words, Sir. Boys are frequently willing to make these generalizations—to arrive at what they believe to be true, but which I believe not to be true. That is the danger; and I wish Mr. Carson would harp on that string a little more.

It is interesting to hear Mr. Radford speak about the definition of a circle. I am reminded of a boy who gave the definition of a circle in words something like the following: "A circle is a round figure bounded by one line: the circumference may be as round as you like, and the diameter may be as broad as you like." (Laughter.) That is a perfectly sound definition of a circle, and I believe you could build up a geometry with that as a definition. I think it illustrates Mr. Radford's point of a boy's ideas of a circle being concerned rather with the curvature than with the centre.

Mr. Carson (replying): In the first place I wish to say how much I welcome the President's repetition of the statement that it is necessary to be definite and precise in regard to each term used in our discussions. The current use of the term intuition is a case in point. Many do not stop to think whether it represents a definite concept to themselves, and the same concept to others, and their thought and outlook on teaching suffer accordingly.

The danger is well shown by the difference between Professor Hobson and myself in the meaning of the word. To him, it means merely the act of perception ; to me, it includes unconscious (and possibly fallacious) generalisations based on perceptions, generalisations which are regarded as material certainties by the mass of men. For our purpose to-day, this latter meaning is that which is under discussion, for it is on these unconsciously formed generalisations that it is proposed to form the first mathematical structure.

Mr. Radford has asked me what I understand by a circle. If he will allow me to say so, the question at issue is not what he or I understand by the term, but what it imports to the child. I entirely agree that the first idea of a circle in the child's mind is uniform or perfect roundness ; but I am equally convinced that, if the finger be placed at the centre of a circle which the child has not seen constructed and he is asked the nearest way from it to the edge, he will say that all ways are equally near. He regards this and other facts, some of which I have mentioned, as essential to a circle, accepting them all with equal faith. To answer Mr. Radford's question specifically, I should say that the circle is to the child an unanalysed body of doctrine, and that all which pertains to its analysis should be deferred to a fitting age.

Mr. Daniell and Mr. Katz have made some allusion to the scope of intuition in connection with mechanics. It is, of course, certain that men who are accustomed to handle heavy loads do get fairly correct intuitive ideas of certain parts of mechanics, but even there they may be mistaken, as I think the speakers hardly realise. Mr. Katz regards their idea that the motion of an isolated body will not endure as a correct intuition, because such bodies as they meet in their experience are not the ideal bodies of mechanics. But experience, viewed completely, points the other way. There is nothing in logic to compel us to go that way ; we *can* adopt any mechanical postulates we please ; but we *should*, in constructing the ideal, follow experience as closely as possible.

The matter can be put in mathematical terminology. We can never demonstrate by experiment that the acceleration of a body tends to zero with the external influence upon the body. But experiment shows that, in every known case without exception, the two become small together, and so we assume that the acceleration *does* so tend to zero, as one basis of our ideal mechanics. The assumption presents no difficulty to those who have even the most informal idea of a limit, but to others it is incomprehensible. Correct ways of thinking engender correct views of nature, and the dim, but widespread, popular idea of a limit has done much to ensure correct generalisations.

Mr. Tuckey has gone to the root of the question raised in my paper by asking how a commom basis of intuitions can be ensured among different members of a class, some of whom must be more developed than others. The same difficulty exists, of course, in a class in any subject. Even if all the members commence together, it will develop, and can only be dealt with by promotion at suitable intervals. We can thus never secure that uniformity for which Mr. Tuckey asks. Nevertheless his question is an important one, and requires an answer.

Admittedly, interdependent spatial intuitions are formed and held by everyone, for the most part subconsciously. Admittedly also, such intuitions can be brought into full consciousness and formalised by some suitable course of training. Now, among all our pupils there must be a maximum set of *common* intuitions which are so formed and can be so developed, and the suggestion is that this set be ascertained and taken as the basis of the first course in geometry, as is already done in analysis. This, I think, is a logical and practical answer to his question.

The **President** called on **Miss Barwell** to read the following paper.

THE ADVISABILITY OF INCLUDING SOME INSTRUCTION IN THE SCHOOL COURSE ON THE HISTORY OF MATHEMATICS.

I HAVE, during the last two years, in addition to my ordinary routine work of preparing candidates for examinations, been giving a series of lectures to the Students of the Training Department of Alexandra College, Dublin. While reading for these lectures, I was greatly struck by the stress laid by Benchara Branford * in his *Study of Mathematical Education* on the importance of what one might call the historical method. He emphasises the fact that the history of each individual development is a brief compendium of the history of the race, and that the sound method of instruction is to let the student travel, in his quest for knowledge, roughly over the same path by which his fathers arrived,—roughly, only, because life is short, and there were quagmires in which our fathers floundered for many centuries.

I thought it would be very good for the training-students to learn a little of how "Mathematics" grew, before they studied how to teach them, and so I sacrificed a certain amount of their very limited time to this object. And I was glad to find how much their interest was stimulated—especially among those who knew a little mathematics,—and though it was barely possible to do more than stimulate interest, one hopes that some of them will care enough to read more of the subject for themselves when the brief fever of training is at an end.

I reproduced and enlarged for them a scheme given in Benchara Branford's book illustrating the development of Mathematical Science, and showing on one side the periods of human history to which its different stages belong, and on the other the corresponding periods of the student's life. This table (Fig. 1) is, very likely, familiar to many here, but as it has been suggested to me that a figure broad at the base and tapering to a vertex does not give quite the right impression, I have ventured to turn it upside down, as being more suggestive of a tree whose beginning is no bigger than a grain of mustard seed, but the spread of whose branches no man shall foretell.

In other classes with girls of sixteen and seventeen, I have also been trying, more informally, to bring in here and there a little of the story of mathematical growth. It could only be done quite casually, as we are much tied down by examinations in Ireland, but it has always aroused very real interest, and called forth a great many questions,—some of them rather posers, but that must, I suppose, be looked upon as a blessing in disguise. One child, after inquiring of the regions that lay beyond her own beat, said, "And after that we can go on and discover more mathematics." I am afraid she will not, but her attitude was encouraging.

There can be no doubt that it is a great gain to the young student, when he can look upon Mathematics as living and growing, rather than as a crystallised thing from a text-book. Does not even a rock appeal more to our imagination when we realise that it has a story? The subject is humanised; it takes a place in the pageant of our race's history. The student begins to take up a right attitude towards it. He realises what it is that makes progress possible,—how the first impulse came from practical need; how ideas can be extended from the purely concrete to the abstract; how necessary it is to have, besides the thought, a compact and adequate means of expressing that thought

* *A Study of Mathematical Education.* Benchara Branford. Clarendon Press, Oxford.

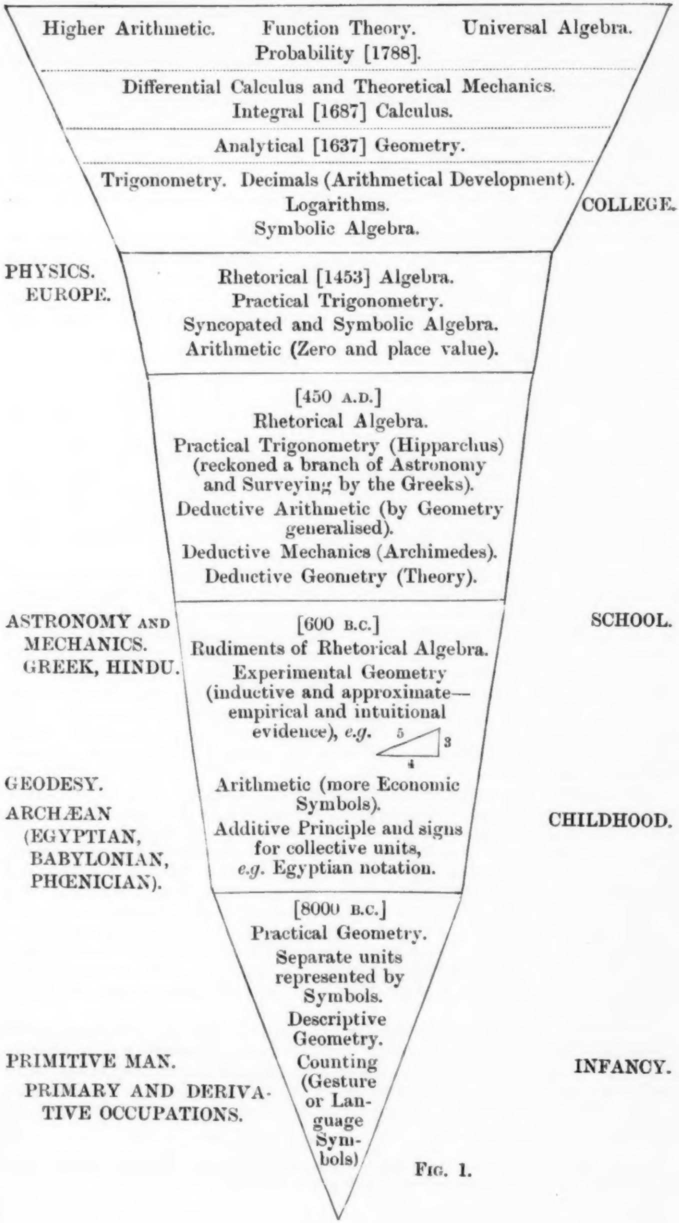
Higher Arithmetic. Function Theory. Universal Algebra.
Probability [1788].
Differential Calculus and Theoretical Mechanics.
Integral [1687] Calculus.
Analytical [1637] Geometry.
Trigonometry. Decimals (Arithmetical Development).
Logarithms.
Symbolic Algebra.
COLLEGE.
PHYSICS.
EUROPE.
Rhetorical [1453] Algebra.
Practical Trigonometry.
Syncopated and Symbolic Algebra.
Arithmetic (Zero and place value).
[450 A.D.]
Rhetorical Algebra.
Practical Trigonometry (Hipparchus) (reckoned a branch of Astronomy and Surveying by the Greeks).
Deductive Arithmetic (by Geometry generalised).
Deductive Mechanics (Archimedes).
Deductive Geometry (Theory).
ASTRONOMY AND MECHANICS.
GREEK, HINDU.
SCHOOL.
[600 B.C.]
Rudiments of Rhetorical Algebra.
Experimental Geometry (inductive and approximate—empirical and intuitional evidence), e.g.
5
3
4
GEODESY.
ARCHÆAN (EGYPTIAN, BABYLONIAN, PHŒNICIAN).
Arithmetic (more Economic Symbols).
Additive Principle and signs for collective units, e.g. Egyptian notation.
CHILDHOOD.
[8000 B.C.]
Practical Geometry.
Separate units represented by Symbols.
Descriptive Geometry.
Counting (Gesture or Language Symbols)
PRIMITIVE MAN.
PRIMARY AND DERIVATIVE OCCUPATIONS.
INFANCY.

Fig. 1.

—the necessity, I mean, of a suitable notation. The mathematician is like an explorer who must have his instruments and commissariat in a portable form if he is to travel far.

No modern schoolboy can appreciate the blessings which he enjoys in the way of notation till he has seen something of the difficulties with which his predecessors had to wrestle. The Egyptian method of writing 21,232,378 will probably give him to think. The Egyptian method (Fig. 2), too, of solving a simple equation by successive guesses, is also very instructive (Fig. 3).

Few things are more amazing than the immense time it has taken for some of the simplest ideas to dawn upon the human mind,—and with what complications people had to struggle, just because the simple idea was not there ! Then the flash of genius came, the epoch-making idea (a commonplace now to the First-form child), came into being, and progress became possible.

Is it not an astonishing thing that in the oldest deciphered work on Mathematics,* a papyrum preserved in the British Museum, copied by the scribe Ahmes from a MS. dating about 2400 B.C., fractions should have been employed with a unit numerator only ($\frac{2}{19}$ being written $\frac{1}{12}+\frac{1}{76}+\frac{1}{114}$), and that in a papyrus found in Upper Egypt, written by a Christian Greek between the fifth and ninth centuries A.D., the very same method of handling fractions should be found ? Nearly three thousand years to take the step from a numerator of one to a numerator of two or three ! That this extreme delay has been largely attributable to the crystallising effect of the examination system of Ancient Egypt only a little diminishes the marvel of it.

What I maintain then is that even a slight knowledge of the outline of the story of Mathematical Science not only stimulates the student's interest, but really gives him a better grasp of his subject. Far too many of our students leave us with far too little an idea of what mathematics are really driving at ; and though, largely thanks to the labours of the Mathematical Association, much of that is changed, it is wonderful how uninquiring our attitude is apt to be towards that which we find set down for us in text-books. The small child is naturally inquisitive, but the middle-aged child takes things for granted rather too readily when they are found in text-books, and does not always wonder quite enough how they came there.

I am continually surprised by the number of pupils I get, at sixteen or seventeen years of age, who have not a notion why the number ten has a special place of importance in our system of numeration. I have frequently been told it is because it can be represented by the symbol 10. It comes to them as quite a refreshing piece of news that it actually has something to do with the human body. And then it pleases them to find that such primitive methods of measuring as those still employed in the British Islands are based on the supposed length of parts of the human body, and that to have a measure one could always carry about with one, and could not mislay, such as a foot, a hand, or a nail, had practical advantages at a time when the need of a more rigid standard was not pressing.

A teacher who is interested in the subject can do much towards widening the horizon of his class, without adding greatly to the burden of things to be learnt. The children's imagination is stirred when they realise something of the long struggle to acquire what to them comes so easily,—the ages it took to learn to count even up to ten ; and that at

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* See *The Teaching of Elementary Mathematics*, by David Eugene Smith. (Macmillan.) From which much of the information in this paper is derived.

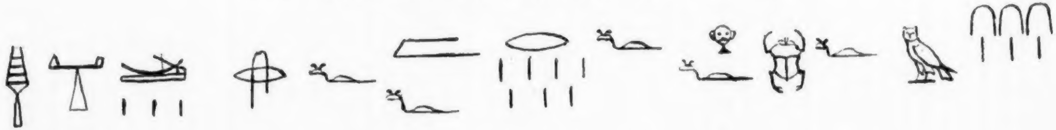
EARLY EGYPTIAN NUMBER SYMBOLS.

2 ten millions + 1 million + 2 hundred thousands + 3 ten thousands + 2 thousands + 3 hundreds + 7 tens + 8 ones
$= 21,232,378$ (additive principle).

FIG. 2.

AN EGYPTIAN EQUATION (ABOUT 2000 B.C.).

$x = 14 + \frac{1}{4} + \frac{1}{97} + \frac{1}{56} + \frac{1}{679} + \frac{1}{776} + \frac{1}{194} + \frac{1}{388}$ is the "root."



$x \quad (\frac{2}{3} + \frac{1}{2} + \frac{1}{7} + 1) = 33.$

FIG. 3.

the present day some savages, and some very small children, still count, "one, two, three——many."

Then comes the struggle to find a notation. The two systems with which the children are already familiar are of course the Roman and the so-called Arabic system; but the simplest is the notched stick. Bakers' men even now, or until quite recently, sometimes leave a tally with a Dublin maidservant, and a notch is cut for every loaf delivered.

The Egyptian system, as is shown in the diagram, had different symbols for different powers of ten. The Babylonians, who wrote on soft bricks, which they afterwards baked, had fewer characters; one for 1, one for 10, one for 100. The Greeks used the first nine letters of the alphabet for the numbers 1 ... 9. After that $\iota=10$, $\kappa=20$, $\lambda=30$, and so on. Thus $\kappa\alpha$ stood for 21.

The difficulty of working these systems will be obvious to any child, and the importance of the discovery of "place-value" will be evident. It is interesting to learn that the so-called Arabic system is really of Hindu origin; that inscriptions were found not fifty years ago in Bombay Presidency (Nana Ghat), probably dating from the third century B.C., in which these numerals appear; that the numerals 4 ... 9 were probably letters of an ancient alphabet. An interesting point is that it was many centuries before the symbol 0 was evolved, and that therefore the place-value element was wanting—its real claim, of course, to superiority. The zero perhaps appeared in the Hindu system about 300 A.D., and the first known use of the symbol occurs in a document of the eighth century A.D.—another instance of the amazing length of time which a simple idea can take to dawn upon humanity.

Whether the Hindus or the Arabs invented the zero symbol is uncertain. It was brought to Europe by an Italian traveller in 1200 A.D. (Leonardo Fibonacci of Pisa), and the Arabs, being traders, helped to disseminate the system; but it was not till about 1500 that it began to get a firm footing in the schools—owing to the invention of printing. The zero was a mere dot, but as so small a symbol could easily be removed by unscrupulous merchants, it was written in the form of a polygon, which easily degenerated into the present symbol.

The history of the rise of geometry emphasises the two facts that theoretical science is first called into being by practical need, and that many facts are known empirically long before they can be scientifically proved. Butcher mentions in his *Greek Studies* that the right-angled property of a triangle whose sides are 3, 4, and 5 was treated by a Chinese Emperor of 1100 B.C. as a piece of magical knowledge, but it was the Greek mind which found out the reason. How the same law was known to the Egyptians, whose "harpedon aptae" or "rope-stretchers" used it in the building of temples, just as the surveyors of the present do, or school-children marking out a tennis-ground; how the periodic rise of the Nile and the consequent obliteration of landmarks brought about an early need for geometrical skill; how in the occult mysteries of astrology the beginnings of a science germinated; how because the Babylonians represented by a circle the year which they computed at 360 days, we still divide a right angle into 90 degrees; how the Egyptian priests measured areas empirically, and kept the results recorded in the temples (making a sacred mystery of the area of a rectangle), and yet Ahmes made a good approximation to the value of π;* how Thales, having earned a fortune by trade, spent it in studying Mathematics from the Egyptian priests, and how his pupil Pythagoras founded the school at Croton and Geometry became a science,—these and many other points of history readily rouse the interest of the young student.

* 3·1605.

Then another point of interest is the rise of Algebra. The Greeks of Euclid's time saw mathematics through the medium of geometry; therefore they did not recognise any power of a variable higher than the third. Euclid, Book II., even when not formally taught, gives many valuable illustrations of algebraic facts; and a class is generally pleased to see Euclid II. 11 worked side by side with a quadratic equation, and to find that the steps—completing the square, and so on—exactly correspond.

About 100 B.C. an advance was made, and Heron of Alexandria spoke boldly of the fourth power of lines, and in solving the quadratic equation "struck" imaginary roots, and five hundred years later the first known work on Algebra was written by Diophantus of Alexandria. He only uses one unknown quantity ς′, the square is Δ^Y (δύναμις) and the cube (K^Y). He goes as far as the sixth power. He solves problems like "Find two numbers whose sum is 20 and the difference of whose squares is 80." He only gives one answer to a quadratic equation, and ignores negative quantities.

A student should surely know that it was the Orient that gave the next important impulse to the science, and that to Bagdad in the eighth century, among other learned men summoned by an enlightened Calif, came the father of modern Algebra, al-Khuwarazmi, whose work *Ilm al-jabr wa'l muqabalah* (the science of redintegration and equation) gives its name to Algebra.

His division of quadratic equations into six classes $ax^2=bx$; $ax^2=c$; $bx=c$; $x^2+bx=c$; $x^2+c=bx$; $x^2=bx+c^2$ is interesting as showing that the simple idea of a general type was not grasped.

I have ventured to take up your time over these scraps of history, familiar as they may be to almost everyone here, because I wanted to bring out some of the points which can hardly fail to interest young students, and which every school-child who has gone any way in Mathematics must surely be the wiser for knowing.

It can hardly be disputed that the child's view of Mathematics must become more intelligent, but the question naturally arises as to how the instruction should be given. Should the history of mathematics be allotted a definite place on the school curriculum, or should the thing be left to the teacher's discretion? There is no doubt that a teacher interested in the subject can quite informally and as occasion arises give all the instruction that is needed; and if all teachers of Mathematics would be at some pains to do so, it would probably be the most satisfactory method, and would save the subject from the risk of becoming that lean and barren thing, a skeleton outline to be got up for examination.

I was greatly interested to receive Mr. Van der Heydn's *Notes on Algebra*, which he kindly sent me after the subject of this paper had been announced, and to find that he has made a special feature of the historical notes,—in which he has introduced, I see, the very same Egyptian equation that I had shown to my training-students, and which is here reproduced. I should like to acknowledge also with gratitude a very kind communication from Professor Genese, in which he tells me that Professor Zeuthen of Copenhagen informed him in 1908 at the Congress at Rome, that he had introduced History of Mathematics as one of the subjects for Degree at Copenhagen.

In a reform of this sort may probably be found the true solution of the problem. If all those who taught Mathematics had had to pass some such examination as that at Copenhagen, the subject would find its way naturally into school work; but until other Universities follow

suit, may one not plead for a place for it in the curriculum of the school ?

In concluding her remarks, **Miss Barwell** said she would like to mention that a great deal of the historical part of her paper had been derived from Smith's *Teaching of Elementary Mathematics*, an American publication which she thought would be of great interest to any teacher.

DISCUSSION ON MISS BARWELL'S PAPER.

The President: I am sure we have all listened with great interest to this paper. There are several things in it which I did not know much about, but I think that probably what Miss Barwell said towards the close of her paper would strike most people as meeting the case best; that is to say, the history of mathematics in schools should be given in small doses interspersed throughout the teaching. In those circumstances it would really enliven and give an interest to the work. I think there is a great danger in any kind of systematic treatment of history at an early stage, when the amount of actual knowledge of mathematics must be very small. In my opinion, if the history goes further than the mathematics which pupils have to learn, they would have no correct understanding of it, and that would be a very barren form of training. Surely the history ought only to keep pace with those aspects of mathematics that are real to the children who are taught. Under those conditions I think it would be of great value in connection with the teaching of arithmetic and algebra and geometry, if occasional indications were given of the way in which the points have historically arisen.

I was very much interested with the long number beginning with millions and the Egyptian notation. It was not so much that I was amused with the quaint animals and so forth, but it struck me that the mathematicians of that time were in some respects a great deal nearer the real thing than the Greeks and Romans were afterwards. They had the symbols according to the powers of ten from left to right. They had no such barbarism as XL. In that way addition and subtraction were not mixed up, and all that they wanted was the one final step of writing one symbol for different things according to its place. But they had the right arrangement, and it struck me that they were just a trifle nearer than the Greeks and Romans were to our actual notation. I had never seen that before, and I must say it was very interesting to me. I will now ask for remarks on the paper.

Miss E. M. Hancock: I thought I should like to say one thing. I noticed that in her paper Miss Barwell made reference to a course which had been introduced into a university abroad. I should like to say that such a course is not unknown here. I do not know whether any students from other universities can give a case, but I know that in my own University of London, those of us who were fortunate enough to work under Professor Harding had an historical course and, though it was not included in the examination for a degree, we did have the advantage of the training.

Miss Margaret Frodsham (Cardiff): I should like to say that I have been very much interested in hearing the paper. Such a course is of very great value to the student in training, for, later, as teacher in the schools, he is able to give some historical outline of the work being done. Even in the earlier stages, if he chooses he may interest the pupils in some particular facts in the historical development of mathematics, but the great point is that the teacher is enabled, through the possession of these facts, to form the school course in a line with

this development, and this is a great gain to the mathematical course in the school.

I know that the discussion on Mr. Carson's paper is already closed, but there is a point which seems to me to link on here. If we take this historical development, and if we consider a point that has not been referred to by its actual name (though Mr. Carson got very near to it in his answers), that is, if we study in this connection the *psychological* development of a child, we shall then find what work should naturally come first, and we shall, by means of this association, feel justified in dealing with the mathematical facts from a practical point of view, and *before* reasons are found; later, by making clear from the standpoint of "development" the necessity for those reasons, we should give the subject an interest which would not otherwise exist.

One particular point which the teacher can illustrate is the historical development of the use of the decimal point. This is given clearly in Mr. B. Branford's book, to which Miss Barwell has referred.

Mr. Plum: I should like to say that for some years I have used what knowledge of the history of mathematics I possess in forms generally, and that I have noticed it has been most useful in keeping up the interest of boys in mathematical work, and has often been the means of inciting an interest in mathematics in the case of boys who ordinarily could not be said to have any great interest in the subject.

I want to ask one question. I did not catch the name of the book from which Miss Barwell said she obtained some of her information. I shall be very glad to have it presently if I may. I am afraid I came in late, so that I heard only a portion of her paper, and in this I was very much interested. May I ask for the name of the book now? I think it was an American publication.

The Chairman: If nobody else wishes to make any remarks on Miss Barwell's paper, I will ask her to reply to those who have spoken.

Miss Barwell (replying): First of all, the names of the books I mentioned are: Benchara Branford's *Study of Mathematical Education* (Clarendon Press); D. E. Smith's *Teaching of Elementary Mathematics* (Macmillan). I think there were not really very many questions asked. I am very much pleased to learn that people are interested in the subject, and, as I am sure will be more and more the case, to know that many of the teachers have already introduced the teaching of this subject into their school work. With regard to the Chairman's remarks as to the introduction of the history of the subject so far as it applies to work which the children have not yet done, I do not think the history should be insisted upon and learnt in that case. I do not think it interests children to know that a thing has been discovered and is there, but visually to show them the growth of mathematics cannot fail to be a good thing. I quite agree that we cannot drill into them the study of things which, from the limits of their knowledge bearing upon them, they could not understand; but it is certainly an advantage to teach them those things that do come within their grasp.

The President: I am asked to call your attention to the fact that there will be at 11.15 to-morrow, in connection with the Association of Public School Science Masters, a discussion on the teaching of Statics, initiated by Mr. Siddons, and a discussion later in the day on the teaching of Dynamics, initiated by Mr. Ashford. All members of the Mathematical Association are cordially invited to come and take part in the discussion.

Dr W. P. Milne read a paper on "The Teaching of Scholarship Mathematics in Secondary Schools."

THE TEACHING OF SCHOLARSHIP MATHEMATICS IN SECONDARY SCHOOLS.

It was considered only natural, a generation or two ago, for the schoolmaster to make it his first business to teach the clever boys, and then to give some attention to his less gifted pupils if he had sufficient time and energy left. Furthermore, a large proportion of what was taught was entirely beyond the comprehension of any but the abler boys. For the last twenty or thirty years a movement has been in progress so to devise schemes of knowledge and methods of imparting it that the rank and file of the pupils in a school shall be able to take full advantage of the instruction given. The result of this psychological study and investigation has been that never before have the imparting of knowledge by the teacher on the one hand and the assimilation of it by the pupil on the other been so well understood as they are at the present day. Though there may be disagreement as to details, there can be no doubt of the great advance made.

On the other hand, there are in every Secondary School some distinctly clever boys—lads of more powerful comprehension and penetrating intelligence than the average pupil. Nothing could be more disastrous to the intellectual well-being of our nation than that the capacities of these students should not be trained up to their highest pitch. They are capable of it, and have therefore the right to it. Some Secondary Schools are unable to give this specialized training for financial reasons. It means a larger staff. But there lurks in the headmaster's mind a hope—or shall we call it an ambition—that some day in the future the funds may be forthcoming which will enable him to deal adequately with his clever boys. The proper training of these able pupils is the most important contribution that our Secondary Schools have it in their power to make towards the maintenance of Great Britain's intellectual position among the nations.

It is needless to remark that the paths followed up by these clever pupils are various and diverse. But a large proportion of them are being trained to compete for scholarships at the Universities, and it is with these we shall deal more particularly here. It has to be noted that for many years past the best pedagogic thought of the country has been directed towards devising methods of teaching the generality of the pupils, while very little has been done to modify or improve the course of instruction given to the scholarship candidate. He has, in fact, been obscured by the clamant needs of the average pupil. The result is that, in the republic of the Secondary School, the status of the scholarship boy was never so low as it is at present, and yet he is the man who for the most part is being trained to lead in the various activities of the nation in his after-life.

Let us assume then that we are dealing with schools whose finances permit them to give special training to the scholarship candidate. The question that next arises is, "What are to be our aims in our teaching of scholarship work?" The answer to that seems to be: There exists a well-defined schedule of mathematical knowledge with sufficiently elastic boundaries which we are expected to impart to scholarship candidates. Within these limits we ought to avail ourselves of the most up-to-date knowledge and methods of the subject. Furthermore, we ought not to teach the subject as though a brick-wall surrounded the schedule, over which no schoolmaster and no schoolboy could possibly climb. The feelings so produced in both teacher and taught are akin to those which a Highland poet has described as existing in his native village before the institution of the annual cheap railway fares to Edinburgh. He sings

of the surprise which his fellow-villagers exhibited when they first discovered for themselves that

> "The warld gangs a' the way tae Edinburgh Toon,
> An' stretches further still."

We ought to teach our pupils that the mathematical world goes all the way to the edge of the schedule and stretches further still.

Again, what is to be the supreme test of the successful teaching of scholarship work? Is it to be whether or not the boy obtains a scholarship at the University? Surely we have passed that stage long ago with regard to the elementary work in the school. Why should we continue to uphold it with regard to the advanced work? It is undoubtedly one test, and a good test, of whether or not a boy possesses a competent knowledge of the range of work prescribed, but it is not the only test. There can be no doubt that if heads of mathematical departments, etc., were to apply the same stringent canons of criticism to the teaching of the scholarship work as they do to the elementary work, a sad tale would be unfolded throughout the country. The same standard of craftsmanship is not demanded by public opinion from the teachers of scholarship work as is required from the teachers of the less advanced subjects.

But some may maintain that this is unnecessary, that the boys being abler can be left more to themselves and learn more by themselves. To a certain extent this is true, but the boys have to learn what they do not learn from the teachers out of the text-book, and we all know that owing to the labour of preparation and the small pecuniary returns, text-books covering the ground of scholarship work are rarely published, and continue to be used when long out of date. Now few boys can rediscover for themselves the theorems of the great masters. They have to be taught them. In fact, the following dictum of Sir William M. Ramsay in a paper on "Classical Research" approximately sums up the situation also with regard to the teaching of the higher mathematics in a Secondary School: "It is the fact that if the young classical researcher is to do good work, half the labour and three-fourths of the thought must come from his trainer. The pupil does the labour for himself, as he thinks, but the trainer also does it, and so is able to keep him in the straight path."

Our object in scholarship teaching ought to be to lay a solid foundation, so that the University teachers may proceed to build at once their superstructure thereon. For that reason the teaching should be sound, so that nothing has to be unlearned when the student proceeds to College. Also, the teaching at School and University should be as far as possible continuous. A careful observation of existing conditions leaves no doubt that much unnecessary dislocation exists when a pupil passes from the Secondary School to the University. I can remember a student who informed me that he had never passed through a time of such intellectual misery in his life as he did in his first term at the University. He had been taught the Calculus at school by what a well-known mathematician has called "the method of pictures and plausibility." He then entered the University, and began straight away the rigorous study of analysis. The formal definition of a limit—so different from what he had been taught at school, what the whole subject was trying to "get at," and above all the sight of the fabric of the Calculus he had learned at school tottering at its base and threatening to crumble to ruin—nearly drove him crazy. He did not know where he was. He dreaded the future, for he had lost confidence in the past. In fact, the abrupt change from the "hail-fellow-well-met" treatment of limits at the schools to the rigorous treatment with

the inevitable ϵ at the Universities was such an insuperable barrier to many students that some years ago analysis bore—and perhaps still bears—amongst students the name of "Epsilonology," and acquaintance with it was regarded as a terrifying experience. The explanation of this phenomenon is simple. The fundamental conceptions of analysis (such as irrational numbers, complex numbers, sequences, etc.) receive no adequate treatment at the Secondary Schools because they will be taught at the Universities, and they receive no adequate treatment at the Universities because they have been taught at the Secondary Schools. Hence they are not explicitly discussed at all. Again, the text-books present the same state of affairs. The elementary text-books in general present the fundamental laws of algebra (the law of signs in multiplication, the laws of commutation, association and distribution, etc.) as though they had dropped from the skies, on the ground that their philosophical explanation is too difficult for some pupils. The more advanced treatises assume that they have been already adequately treated in the elementary books. The only book with which I am acquainted that attempts to present the philosophical side of the fundamental laws *so as to be understood by scholarship candidates* is the *Arithmétique* of Jules Tannery, the great French mathematician. To give an illustration of the lacuna that exists at this stage of the pupil's mathematical training, I may quote from the preface of the *Cours d'Analyse* by Goursat:

"Mathematical Analysis being essentially the science of the continuous, it seems that logically every course of Analysis ought to commence with the study of irrational numbers. I have, however, supposed this notion acquired. The theory of incommensurables is treated in excellent and well-known works, and in such a masterly manner that it seemed to me unnecessary to go over the ground again." Now every student knows that Goursat is just the book he wants on Analysis at the beginning of his University career, but its usefulness is sadly impaired by the fact that our schoolboys are not, in general, being prepared on up-to-date lines so as to be able to appreciate it. We still teach with great dignity and detail the method of finding the greatest term in the Binomial Expansion, while it is no exaggeration to say that not fifty per cent. of the candidates who could write out a faultless essay on the above theorem have a clear notion of the difference between a rational and an irrational number, or could quote the simplest instance of a property possessed by irrationals which is not also possessed by the rationals.

When any state of affairs is unsatisfactory in the teaching world, it is customary to throw the blame on the external examiner. In the case of scholarship work the external examiner cannot in most cases be blamed. He has ceased to regard the manipulative faculty as the sole test of mathematical proficiency. He has discarded artificial questions, and now sets natural ones; he sets essay papers to test whether the candidates have enquired into or even thought about the fundamental ideas of the subject, whether they can take a comprehensive view of a branch of mathematics and show in essay form the main theorems, and what they are each and all "getting at." What more can we teachers desire? What now remains for those who work in Secondary Schools is to bring the teaching more into line with the most up-to-date trend of mathematical thought. We want *reform* but not *revolution*. We want improvement in the various portions of the subject which we teach, but we want still more to improve the spirit of the whole schedule of knowledge which we are called upon to impart. Thus Theory of Functions looms large in the future of our pupils: is it not imperative that the teaching of algebra should be moulded towards that end? Is the Theory of Probability and

Diophantine Equations essential knowledge on the part of a pupil who does not know what an irrational number is?

The following are one or two points which are worthy of consideration:

(1) A celebrated mathematician—Salmon I think it was—has said that if a tithe of the time now spent in finding the areas of dreadful circles inscribed to dreadful triangles self-conjugate to dreadful conics were devoted to the application of that complicated analysis which is required for Cubic Curves, the pupils would receive the same drill in manipulative analysis and, in addition, a knowledge of more geometry than they at present possess. I have no wish to advocate the inclusion of Higher Plane Curves into the scholarship syllabus, but the *principle* contained in the above pronouncement ought to have our whole-hearted sympathy. There are many really important developments in the geometry of conics that required high manipulative faculty, such as the theory of the F and Φ conics and many loci connected therewith. The geometry of the triangle and its circles offers a large field of investigation on which our pupils could practise harder analysis and learn something more about geometry at the same time.

(2) Another noted mathematician has said that it is easier to learn the Calculus than methods of avoiding it. We all remember what a cumbrous business it used to be to find the centre of pressure of a triangular lamina, etc., by means of infinite series. Again, the proofs of the "Potential Energy$=\frac{1}{2}\Sigma EV$" in Sir J. J. Thomson's *Elements of the Mathematical Theory of Electricity and Magnetism* is at least as hard on its algebraic side as it is on its physical, inasmuch as the prevailing state of mathematical tradition at the time when he wrote the book precluded a whole-hearted use of the Calculus. That is all gone now, but many disabilities of the same kind still remain. Our scholarship candidates actually use in many instances harder methods of applying a given formula than would the experts in the subject in dealing with the same formula. One has only to take the case of summation-diagrams. The application of graphical methods to the proofs of those *bêtes-noires* of our student days, the Binomial, Exponential, and Logarithmic Expansions, renders their rigorous demonstrations much more easy, as the pupil sees laid out before his eyes in panoramic form the numbers with which he is dealing. No one would nowadays, when cartridges have been invented, dream of sending forth an army armed with muzzle-loading guns and powder flasks and funnels and ram-rods. Why do we use "flintlock" methods in mathematics?

(3) The choice of a clear notation is of importance in our mathematical text-books. Several of our learned societies, in order to cheapen the cost of publication, have been compelled to simplify, from the printer's point of view, the notation used. (Thus it is easier to print p/q for $\frac{p}{q}$ and $n!$ for $\underline{|n}$.) But though this form of notation presents no obstacle to mature experts in the case of the learned societies, it is a very different matter with the scholarship candidates in our Secondary Schools. What is a simplified notation for the printer is not generally a simplified notation for the student. We need only quote the case of Chrystal's *Algebra*. The usefulness of that great book has been greatly diminished by the notation used. It is often a matter of considerable difficulty to comprehend at a glance the import of a formula in that work, and though the student may struggle once to see what a complicated expression means, he cannot be expected frequently to interrupt his study of a difficult argument to master mere difficulties of the eye. The interests of the young student must not be sacrificed to those of the printer.

(4) Again, more attention should be paid to teaching fundamental ideas. Things are better in this respect than they used to be, as may be seen from a historical glance at the teaching of "infinity" and "imaginaries" in geometry. A man who in his day had been one of the first six wranglers found that in his declining years he had to teach about this new thing called "infinity." He said quite frankly, "Things at Infinity are bad enough, but when you get imaginary things stowed away at Infinity, it beats my comprehension." Teachers of the last generation had clearer notions on Infinity, and taught that "There is a happy land, far, far away," and its name is Infinity. But the idea had crystallized into a definite shape when a noted Cambridge teacher defined the imaginary Circular Points at Infinity as the two old extinct windmills at Shelford and Trumpington respectively. The designation of these points by I and J and their subsequent baptism as Isaac and Jacob mark the final stage in the evolution of the accurate teaching about these once mysterious Circular Points at Infinity. It behoves teachers, therefore, to introduce the fundamentals of mathematics to our pupils in a less haphazard way.

(5) I have frequently found the following method of teaching a subject useful. Instead of asking the pupils to read the bookwork, I cyclostyle papers containing the bookwork broken up into easy problems, each following after and depending on the one in front. In this way the pupil discovers a good deal of the bookwork for himself, and is stimulated and exhilarated thereby.

That many teachers are dissatisfied with the prevailing state of affairs with regard to the teaching of scholarship work cannot be doubted. Thus I had a note from Mr. W. J. Greenstreet this term deploring the fact that tangential coordinates receive such scanty attention throughout the Secondary Schools of this country, while I had recently a long letter from Mr. G. W. Palmer referring to the chaotic state of the teaching of complex numbers, infinite series, etc. My suggestion then is this. Would it not be desirable that the Mathematical Association should appoint a committee to go into the whole question of scholarship teaching, and make a full report thereon? Surely it is the proper body to take in hand a most important, pressing, and difficult piece of work.

The President invited remarks on Dr. Milne's paper.

The Rev. E. W. Barnes: I should like to say that I have seen the work of a fairly large number of scholarship candidates soon after they have come up to the universities in recent years, and that I think Dr. Milne takes an unduly pessimistic view of the present situation. I am inclined to think that not only have the text-books improved, and that there has been a marked advance in accuracy and range during the last twenty years, but also that the teaching of the candidates at the schools has steadily improved, so that now the boys come up, many of them knowing accurately a number of things which are surprising to those of us at Cambridge, who know the labour which must be expended before such success is gained. Thus, my own impression is quite different from that which the present situation appears to have left on Dr. Milne's mind.

The President: I should like to corroborate what Dr. Barnes said. No doubt some of the books at the universities, when the new methods were introduced, were ahead of the schools. But that was inevitable. It took some time before the new ideas were taught in the schools, because the teachers had been brought up themselves among the old ideas of teaching the calculus and so forth. To expect a completely different standard in methods of teaching in a short time was to expect

the accomplishment of a very difficult thing. But now a sufficient number of years have passed for a new generation of teachers to have arisen, and these have actually succeeded in doing a great deal towards introducing the class of boy under discussion to newer ideas. I have noticed during the thirty or more years in which I taught general mathematics at Cambridge that there was a regular improvement in the conceptions of mechanics on the part of boys who came from school. In the early part of that time they were brought up on old books such as Parkinson's *Mechanics*, but as soon as Dr. Garnett's book appeared a great change was produced. At first things moved somewhat slowly, but within ten or fifteen years there was a very marked increase in the number of boys who really did know something about the principles of mechanics, especially as to what the quantities involved were. I noticed during the whole of those years that there was a considerable steady, though slow, improvement in that way, and I attribute it mainly to the change that was brought into the schools by a new generation of teachers who had learnt something at the universities, and who brought these new ideas gradually to bear on their school teaching; and I imagine that, as regards the foundation of pure mathematics, the calculus in particular, a good deal of such influence must be seen in that case too. In fact, one has only to be at the meetings of this Association to become aware of the fact that there are a considerable number of teachers of mathematics in the schools who are fully acquainted with all the modern ideas in connection with these subjects, for instance, that the doctrine of limits must in some form or other be placed at the base of all treatments of the calculus. That is fully appreciated by all in this Society who take part in the discussions, so that we may take it that something has been gained. No doubt there are schools, and there are teachers in schools, that still lag behind; but they are being gradually eliminated, and I think we may look forward in the hope that they will be still more fully eliminated in the future.

I was very glad to hear the importance emphasised of attention to this class of boys in schools. It is, I feel as strongly as any one, our main business to improve the teaching of what I venture to call the intellectual democracy. That is a most important task which ought not to be neglected. But at the same time we have got to remember that the torch of science has been, and always will be, carried forward by the few and not by the many, and it is of the greatest importance, as was emphasised by the speaker, that we should not, in our enthusiasm for dealing properly with the rank and file, go so far as to allow the teaching of the best class of student to deteriorate. I think there is some possibility of their being partly, if not wholly, neglected, while we are devoting our attention to the teaching of the great body of students. I think it is always difficult to keep one's attention on two things at the same time. Whatever attracts one's attention for the time being seems the most important. We have to remember that there are other things just as important, which we must not neglect. We have had to attend of late years to the teaching of the average boy and girl, but we must not forget that there are others who do not belong to the average, and that their teaching must not be neglected. I was very pleased indeed to hear that point emphasised by Dr. Milne.

The President announced that Mr. Siddons would give a brief account of the work of the Committee.

Mr. Siddons: This has only just been sprung on me at a moment's notice. I think members of the Association will be interested to hear that the Committee are not merely sleeping. The Special Committees have all got to work, and they are considering, though they have not yet

published, Reports on the general curricula in the particular kind of schools with which they are concerned. The General Committee has had a very busy time in connection with examination papers and examination schedules. I may say that within the last month or two, two bodies have received, or are just going to receive, our proposals. I think it is a thing that we may be distinctly pleased about—that two important examining bodies are willing to consider our suggestions. Further than that, one important examining body proposed to set a new paper in the course of last year, and they very kindly listened to many suggestions that we had to make in regard to a proposed specimen paper, and when the specimen paper was published, it was seen that they had made many changes in the paper in accordance with our recommendations.

We are doing what we can to keep in touch with the examining bodies, and hope to do still more in the next twelve months. I say this request is only sprung upon me at a moment's notice, and I can only mention these matters to you to let you know that the Committee is doing what it can, but at present it has nothing to put before you in the way of a formal report.

Mr. Jackson: May I ask a question in connection with this? I want to ask whether it is not a fact that, owing to the exigencies of time, it has not been possible for the Committee to give adequate consideration to these matters, and whether the communications have yet taken place between the Executive Committee and the examining bodies?

Mr. Siddons: In one case, which has already been sent in, action was taken on instructions received from the General Committee. The General Committee did not actually pass the draft in its final form. There was not time. The examining body did not give us long enough. In the other case our recommendations have not been sent in. I used the word "or are just going to receive our proposals." The Committee is meeting to-morrow to consider the final form, and what is proposed for consideration is already before the members of the General Committee and will be finally settled at a meeting to-morrow.

Mr. Siddons (in proposing a vote of thanks to the President) said: I propose that a very hearty vote of thanks of this Association be accorded to Professor Hobson. He has been very kind to us. He is a very busy man, but he has found time to do a great deal to help and encourage us in the course of the last two years, not least by the kind words he has just said.

The vote of thanks having been passed unanimously,

The President said: In reply to your very kind vote of thanks, I have to say that it has been a very great pleasure to me during the last two years to be President of this Association, because I have had an opportunity of meeting people who perhaps have taken a wider view in many ways and who, at all events, have had an experience which differs from the experience of those I am most in the habit of meeting. I have learnt a very great deal by listening to what has been said at the meetings of this body, and I shall always look back to the two years that I spent in the chair with very great satisfaction. I feel that the experience has widened my outlook, and that I have heard the points of view of those who come in contact with a larger class of students than I have ever been brought into touch with myself. I shall always feel grateful to the Society for having given me an opportunity of getting an extended outlook on mathematical education in the country generally.

MATHEMATICAL ASSOCIATION.

LONDON BRANCH.

THE annual meeting of the branch was held at the Polytechnic, Regent Street, on Feb. 8th.

Dr. A. N. Whitehead, F.R.S., was elected President and delivered the address printed below. A vote of thanks was passed to the retiring President, Prof. M. J. M. Hill, F.R.S.

Mr. Abbott was re-elected Chairman; Miss D. Catmur was added to the committee, and the former officers and committee were re-elected.

Mr. J. Katz, B.A., Borough School, Croydon, read a paper on "The Use of the History of Mathematics in teaching Elementary Mathematics."

PRESIDENTIAL ADDRESS TO THE LONDON BRANCH OF THE MATHEMATICAL ASSOCIATION.

THE situation in regard to education at the present time cannot find its parallel without going back for some centuries to the break up of the medieval traditions of learning. Then, as now, the traditional intellectual outlook, despite the authority which it had justly acquired from its notable triumphs, had grown to be too narrow for the interests of mankind. The result of this shifting of human interest was a demand for a parallel shifting of the basis of education, so as to fit the pupils for the ideas which later in life would in fact occupy their minds. Any serious fundamental change in the intellectual outlook of human society must necessarily be followed by an educational revolution. It may be delayed for a generation by vested interests or by the passionate attachment of some leaders of thought to the cycle of ideas within which they received their own mental stimulus at an impressionable age. But the law is inexorable that education to be living and effective must be directed to informing pupils with ideas, and to creating for them capacities which will enable them to appreciate the current thought of the epoch.

There is no such thing as a successful system of education in a vacuum, that is to say, a system which is divorced from immediate contact with the existing intellectual atmosphere. Education which is not modern shares the fate of all organic things which are kept too long.

But the blessed word "modern" does not really solve our difficulties. What we mean is, "relevant to modern thought, either in the ideas imparted or in the aptitudes produced." Something found out only yesterday may not really be modern in this sense. It may belong to some bygone system of thought prevalent in a previous age, or, what is very much more likely, it may be too recondite. When we demand that education should be relevant to modern thought, we are referring to thoughts broadly spread throughout cultivated society. It is this question of the unfitness of recondite subjects for use in general education which I wish to make the keynote of my address this afternoon.

It is in fact rather a delicate subject for us mathematicians. Outsiders are apt to accuse our subject of being recondite. Let us grasp the nettle at once and frankly admit that in general opinion it is the very typical example of reconditeness. By this word I do not mean difficulty, but that the ideas involved are of highly special application, and rarely influence thought.

This liability to reconditeness is the characteristic evil which is apt to destroy the utility of mathematics in liberal education. So far as it clings to the educational use of the subject, so far we must acquiesce in a miserably low level of mathematical attainment among cultivated

people in general. I yield to no one in my anxiety to increase the educational scope of mathematics. The way to achieve this end is not by a mere blind demand for more mathematics. We must face the real difficulty which obstructs its extended use.

Is the subject recondite? Now, viewed as a whole, I think it is. *Securus judicat orbis terrarum*—the general judgment of mankind is sure.

The subject as it exists in the minds and in the books of students of mathematics *is* recondite. It proceeds by deducing innumerable special results from general ideas, each result more recondite than the preceding. It is not my task this afternoon to defend mathematics as a subject for profound study. It can very well take care of itself. What I want to emphasise is, that the very reasons which make this science a delight to its students are reasons which obstruct its use as an educational instrument—namely, the boundless wealth of deductions from the interplay of general theorems, their complication, their apparent remoteness from the ideas from which the argument started, the variety of methods, and their purely abstract character, which brings, as its gift, eternal truth.

Of course, all these characteristics are of priceless value to students; for ages they have fascinated some of the keenest intellects. My only remark is that, except for a highly selected class, they are fatal in education. The pupils are bewildered by a multiplicity of detail, without apparent relevance either to great ideas or to ordinary thoughts. The extension of this sort of training in the direction of acquiring more detail is the last measure to be desired in the interests of education.

The conclusion at which we arrive is that mathematics, if it is to be used in general education, must be subjected to a rigorous process of selection and adaptation. I do not mean, what is of course obvious, that however much time we devote to the subject the average pupil will not get very far. But that, however limited the progress, certain characteristics of the subject, natural at any stage, must be rigorously excluded. The science as presented to young pupils must lose its aspect of reconditeness. It must, on the face of it, deal directly and simply with a few general ideas of far-reaching importance.

Now, in this matter of the reform of mathematical instruction, the present generation of teachers may take a very legitimate pride in its achievements. It has shown immense energy in reform, and has accomplished more than would have been thought possible in so short a time. It is not always recognised how difficult is the task of changing a well-established curriculum entrenched behind public examinations.

But for all that, great progress has been made, and, to put the matter at its lowest, the old dead tradition has been broken up. I want to indicate this afternoon the guiding idea which should direct our efforts at reconstruction. I have already summed it up in a phrase. "We must aim at the elimination of reconditeness from the educational use of the subject."

Our courses of instruction should be planned to illustrate simply a succession of ideas of obvious importance. All pretty divagations should be rigorously excluded. The goal to be aimed at is that the pupil should acquire familiarity with abstract thought, should realise how it applies to particular concrete circumstances, and should know how to apply general methods to its logical investigation. With this educational ideal nothing can be worse than the aimless accretion of theorems in our text-books, which acquire their position merely because the children can be made to learn them and examiners can set neat questions on them. The bookwork to be learnt should all be very important as illustrating ideas. The examples set—and let there be as many examples as teachers find necessary—should be direct illustra-

tions of the theorems, either by way of abstract particular cases or by way of application to concrete phenomena. Here it is worth remarking that it is quite useless to simplify the bookwork, if the examples set in examinations in fact require an extended knowledge of recondite details. There is a mistaken idea that problems test ability and genius, and that bookwork tests cram. This is not my experience. Only boys who have been specially crammed for scholarships can ever do a problem paper successfully. Bookwork properly set, not in mere snippets according to the usual bad plan, is a far better test of ability, provided that it is supplemented by direct examples. But this is a digression on the bad influence of examinations on teaching.

The main ideas which lie at the base of mathematics are not at all recondite. They are abstract. But one of the main objects of the inclusion of mathematics in a liberal education is to train the pupils to handle abstract ideas. The science constitutes the first large group of abstract ideas which naturally occur to the mind in any precise form. For the purposes of education, mathematics consists of the relations of number, the relations of quantity, and the relations of space. This is not a general definition of mathematics, which, in my opinion, is a much more general science. But we are now discussing the use of mathematics in education. These three groups of relations, concerning number, quantity, and space, are interconnected.

Now, in education we proceed from the particular to the general. Accordingly, children should be taught the use of these ideas by practice among simple examples. My point is this:—The goal should be, not an aimless accumulation of special mathematical theorems, but the final recognition that the preceding years of work have illustrated those relations of number, and of quantity, and of space, which are of fundamental importance. Such a training should lie at the base of all philosophical thought. In fact elementary mathematics rightly conceived would give just that philosophical discipline of which the ordinary mind is capable. But what at all costs we ought to avoid, is the pointless accumulation of details. As many examples as you like; let the children work at them for terms, or for years. But these examples should be direct illustrations of the main ideas. In this way, and that only, can the fatal reconditeness be avoided.

I am not now speaking in particular of those who are to be professional mathematicians, or of those who for professional reasons require a knowledge of certain mathematical details. We are considering the liberal education of all students, including these two classes. This general use of mathematics should be the simple study of a few general truths, well illustrated by practical examples. This study should be conceived by itself, and completely separated in idea from the professional study mentioned above, for which it would make a most excellent preparation. Its final stage should be the recognition of the general truths which the work done has illustrated. As far as I can make out, at present the final stage is the proof of some property of circles connected with triangles. Such properties are immensely interesting to mathematicians. But are they not rather recondite, and, what is the precise relation of such theorems to the ideal of a liberal education? The end of all the grammatical studies of the student in classics is to read Virgil and Horace—the greatest thoughts of the greatest men. Are we content, when pleading for the adequate representation in education of our own science, to say that the end of a mathematical training is that the student should know the properties of the nine-point circle? I ask you frankly, is it not rather a "come down"?

This generation of mathematical teachers has done so much strenuous

work in the way of reorganising mathematical instruction that there is no need to despair of its being able to elaborate a curriculum which shall leave in the minds of the pupils something even nobler than "the ambiguous case."

Let us think how this final review, closing the elementary course, might be conducted for the more intelligent pupils. Partly no doubt it requires a general oversight of the whole work done, considered without undue detail so as to emphasise the general ideas used, and their possibilities of importance when subjected to further study. Also the analytical and geometrical ideas find immediate application in the physical laboratory where a course of simple experimental mechanics should have been worked through. Here the point of view is twofold, the physical ideas and the mathematical ideas illustrate each other.

The mathematical ideas are essential to the precise formulation of the mechanical laws. The idea of a precise law of nature, the extent to which such laws are in fact verified in our experience, and the rôle of abstract thought in their formulation, then become practically apparent to the pupil. The whole topic of course requires detailed development with full particular illustration, and is not suggested as requiring merely a few bare abstract statements.

It would, however, be a grave error to put too much emphasis on the mere process of direct explanation of the previous work by way of final review. My point is, that the latter end of the course should be so selected that in fact the general ideas underlying all the previous mathematical work should be brought into prominence. This may well be done by apparently entering on a new subject. For example, the ideas of quantity and the ideas of number are fundamental to all precise thought. In the previous stages they will not have been sharply separated; and children are, rightly enough, pushed on to algebra without too much bother about number and quantity. But the more intelligent among them at the end of their curriculum would gain immensely by a careful consideration of those fundamental properties of quantity in general which lead to the introduction of numerical measurement. This is a topic which also has the advantage that the necessary books are actually to hand. Euclid's fifth book is regarded by those qualified to judge as one of the triumphs of Greek mathematics. It deals with this very point. Nothing can be more characteristic of the hopelessly illiberal character of the traditional mathematical education than the fact that this book has always been omitted. It deals with ideas, and therefore was ostracised. Of course a careful selection of the more important propositions and a careful revision of the argument are required. This also is to hand in the publications of my immediate predecessor in the office of president, Prof. Hill. Furthermore, in Sir T. L. Heath's complete edition of Euclid, there is a full commentary embodying most of what has been said and thought on the point. Thus it is perfectly easy for teachers to inform themselves generally on the topic. The whole book would not be wanted, but just the few propositions which embody the fundamental ideas. The subject is not fit for backward pupils; but certainly it could be made interesting to the more advanced class. There would be great scope for interesting discussion as to the nature of quantity, and the tests which we should apply to ascertain when we are dealing with quantities. The work would not be at all in the air, but would be illustrated at every stage by reference to actual examples of cases where the quantitative character is absent, or obscure, or doubtful, or evident. Temperature, heat, electricity, pleasure and pain, mass and distance could all be considered.

Another idea which requires illustration is that of functionality. A

function in analysis is the counterpart of a law in the physical universe, and of a curve in geometry. Children have studied the relations of functions to curves from the first beginning of their study of algebra, namely in drawing graphs. Of recent years there has been a great reform in respect to graphs. But at its present stage it has either gone too far or not far enough. It is not enough merely to draw a graph. The idea behind the graph—like the man behind the gun—is essential in order to make it effective. At present there is some tendency merely to set the children to draw curves, and there to leave the whole question.

In the study of simple algebraic functions and of trigonometrical functions we are initiating the study of the precise expression of physical laws. Curves are another way of representing these laws. The simple fundamental laws—such as the inverse square and the direct distance—should be passed under review, and the applications of the simple functions to express important concrete cases of physical laws considered. I cannot help thinking that the final review of this topic might well take the form of a study of some of the main ideas of the differential calculus applied to simple curves. There is nothing particularly difficult about the conception of a rate of change; and the differentiation of a few powers of x, such as x^2, x^3, etc., could easily be effected; perhaps by the aid of geometry even $\sin x$ and $\cos x$ could be differentiated. If we once abandon our fatal habit of cramming the children with theorems which they do not understand and will never use, there will be plenty of time to concentrate their attention on really important topics. We can give them familiarity with conceptions which really influence thought.

Before leaving this topic of Physical Laws and mathematical functions, there are other points to be noticed. The fact that the precise law is never really verified by observation in its full precision is capable of easy illustration and of affording excellent examples. Again, statistical laws, namely laws which are only satisfied on the average by large numbers, can easily be studied and illustrated. In fact a slight study of statistical methods and their application to social phenomena affords one of the simplest examples of the application of algebraic ideas.

Another way in which the students' ideas can be generalised is by the use of the History of Mathematics, conceived not as a mere assemblage of the dates and names of men, but as an exposition of the general current of thought which occasioned the subjects to be objects of interest at the time of their first elaboration. The use of the History of Mathematics is to be considered at a later stage of our proceedings this afternoon. Accordingly I merely draw attention to it now, to point out that perhaps it is the very subject which may best obtain the results for which I am pleading.

We have indicated two main topics, namely general ideas of quantity and of Laws of Nature, which should be an object of study in the mathematical curriculum of a liberal education. But there is another side to mathematics which must not be overlooked. It is the chief instrument for discipline in logical method.

Now, what is logical method, and how can anyone be trained in it ?

Logical method is more than the mere knowledge of valid types of reasoning and practice in the concentration of mind necessary to follow them. If it were only this, it would still be very important; for the human mind was not evolved in the bygone ages for the sake of reasoning, but merely to enable mankind with more art to hunt between meals for fresh food supplies. Accordingly few people can follow close reasoning without considerable practice.

More than this is wanted to make a good reasoner, or even to enlighten ordinary people with knowledge of what constitutes the essence of the

art. The art of reasoning consists in getting hold of the subject at the right end, of seizing on the few general ideas which illuminate the whole, and of persistently marshalling all subsidiary facts round them. Nobody can be a good reasoner unless by constant practice he has realised the importance of getting hold of the big ideas and of hanging on to them like grim death. For this sort of training geometry is, I think, better than algebra. The field of thought of algebra is rather obscure, whereas space is an obvious insistent thing evident to all. Then the process of simplification, or abstraction, by which all irrelevant properties of matter, such as colour, taste, and weight, are put aside is an education in itself. Again, the definitions and the propositions assumed without proof illustrate the necessity of forming clear notions of the fundamental facts of the subject matter and of the relations between them. All this belongs to the mere prolegomena of the subject. When we come to its development, its excellence increases. The learner is not initially confronted with any symbolism which bothers the memory by its rules, however simple they may be. Also, from the very beginning the reasoning, if properly conducted, is dominated by well-marked ideas which guide each stage of development. Accordingly the essence of logical method receives immediate exemplification.

Let us now put aside for the moment the limitations introduced by the dullness of average pupils and the pressure on time due to other subjects, and consider what geometry has to offer in the way of a liberal education. I will indicate some stages in the subject, without meaning that necessarily they are to be studied in this exclusive order. The first stage is the study of *congruence*. Our perception of congruence is in practice dependent on our judgments of the invariability of the intrinsic properties of bodies when their external circumstances are varying. But however it arises, congruence is in essence the correlation of two regions of space, point by point, so that all homologous distances and all homologous angles are equal. It is to be noticed that the definition of the equality of lengths and angles is their congruence, and all tests of equality, such as the use of the yard measure, are merely devices for making immediate judgments of congruence easy. I make these remarks to suggest that apart from the reasoning connected with it, congruence, both as an example of a larger and very far-reaching idea and also for its own sake, is well worthy of attentive consideration. The propositions concerning it elucidate the elementary properties of the triangle, the parallelogram, and the circle, and of the relations of two planes to each other. It is very desirable to restrict the proved propositions of this part within the narrowest bounds, partly by assuming redundant axiomatic propositions, and partly by introducing only those propositions of absolutely fundamental importance.

The second stage is the study of similarity. This can be reduced to three or four fundamental propositions. Similarity is an enlargement of the idea of congruence, and, like that idea, is another example of a one-to-one correlation of points of spaces. Any extension of study of this subject might well be in the direction of the investigation of one or two simple properties of similar and similarly situated rectilinear figures. The whole subject receives its immediate applications in plans and maps. It is important, however, to remember that trigonometry is really the method by which the main theorems are made available for use.

The third stage is the study of the elements of trigonometry. This is the study of the periodicity introduced by rotation and of properties preserved in a correlation of similar figures. Here for the first time we introduce a slight use of the algebraic analysis founded on the study of number and quantity. The importance of the periodic char-

acter of the functions requires full illustration. The simplest properties of the functions are the only ones required for the solution of triangles, and the consequent applications to surveying. The wealth of formulæ, often important in themselves, but entirely useless for this type of study, which crowd our books should be rigorously excluded, except so far as they are capable of being proved by the pupils as direct examples of the bookwork.

This question of the exclusion of formulæ is best illustrated by considering this example of Trigonometry, though of course I may well have hit on an unfortunate case in which my judgment is at fault. A great part of the educational advantage of the subject can be obtained by confining study to Trigonometry of one angle and by exclusion of the addition formulæ for the sine and cosine of the sum of two angles. The functions can be graphed, and the solution of triangles effected. Thus the aspects of the science as (1) embodying analytically the immediate results of some of the theorems deduced from congruence and similarity, (2) as a solution of the main problem of surveying, (3) as a study of the fundamental functions required to express periodicity and wave motion will all be impressed on the pupils' minds both by bookwork and example.

If it be desired to extend this course, the addition formulæ should be added. But great care should be taken to exclude specialising the pupils in the wealth of formulæ which comes in their train. By "exclude" is meant that the pupils should not have spent time or energy in acquiring any facility in their deduction. The teacher may find it interesting to work a few such examples before a class. But such results are not among those which learners need retain. Also, I would exclude the whole subject of circumscribed and inscribed circles both from Trigonometry and from the previous geometrical courses. It is all very pretty, but I do not understand what its function is in an elementary non-professional curriculum.

Accordingly, the actual bookwork of the subject is reduced to very manageable proportions. I was told the other day of an American college where the students are expected to know by heart 90 formulæ or results in Trigonometry alone. We are not quite so bad as that. In fact, in Trigonometry we have nearly approached the ideal here sketched out as far as our elementary courses are concerned.

The fourth stage introduces Analytical Geometry. The study of Graphs in algebra has already employed the fundamental notions, and all that is now required is a rigorously pruned course on the straight line, the circle, and the three types of conic sections, defined by the forms of their equations. At this point there are two remarks to be made. It is often desirable to give our pupils mathematical information which we do not prove. For example, in co-ordinate geometry, the reduction of the general equation of the second degree is probably beyond the capacities of most of the type of students whom we are considering. But that need not prevent us from explaining the fundamental position of conics, as exhausting the possible types of such curves.

The second remark is to advocate the entire sweeping away of geometrical conics as a separate subject. Naturally, on suitable occasions the analysis of analytical geometry will be lightened by the use of direct deduction from some simple figure. But geometrical conics, as developed from the definition of a conic section by the focus and directrix property, suffers from glaring defects. It is hopelessly recondite. The fundamental definition of a conic, $SP=e.PM$, usual in this subject at this stage, is thoroughly bad. It is very recondite, and has no obvious importance. Why should such curves be studied at all, any

more than those defined by an indefinite number of other formulæ? But when we have commenced the study of the Cartesian Methods, the equations of the first and second degrees are naturally the first things to think about.

In this ideal course of Geometry, the fifth stage is occupied with the elements of Projective Geometry. The general ideas of cross ratio and of projection are here fundamental. Projection is yet a more general instance of that one-to-one correlation which we have already considered under congruence and similarity. Here again we must avoid the danger of being led into a bewildering wealth of detail.

The intellectual idea which projective geometry is to illustrate is the importance in reasoning of the correlation of all cases which can be proved to possess in common certain identical properties. The preservation of the projective properties in projection is the one important educational idea of the subject. Cross ratio enters as the fundamental metrical property which is preserved. The few propositions considered are selected to illustrate the two allied processes which are made possible by this procedure. One is proof by simplification. Here the simplification is psychological and not logical—for the general case is logically the simplest. What is meant is:—Proof by considering the case which is in fact the most familiar to us, or the easiest to think about. The other procedure is the deduction of particular cases from general truths as soon as we have a means of discovering such cases or a criterion for testing them.

The projective definition of conic sections and the identity of the results obtained with the curves derived from the general equation of the second degree are capable of simple exposition, but lie on the border-line of the subject. It is the sort of topic on which information can be given, and the proofs suppressed.

The course of geometry as here conceived in its complete ideal—and ideals can never be realised—is not a long one. The actual amount of mathematical deduction at each stage in the form of bookwork is very slight. But much more explanation would be given, the importance of each proposition being illustrated by examples, either worked out or for students to work, so selected as to indicate the fields of thought to which it applies. By such a course the student would gain an analysis of the leading properties of space, and of the chief methods by which they are investigated.

The study of the elements of mathematics, conceived in this spirit, would constitute a training in logical method together with an acquisition of the precise ideas which lie at the base of the scientific and philosophical investigation of the universe. Would it be easy to continue the excellent reforms in mathematical instruction, which this generation has already achieved, so as to include in the curriculum this wider and more philosophic spirit? Frankly, I think that this result would be very hard to achieve as the result of single individual efforts. For reasons which I have already briefly indicated, all reforms in education are very difficult to effect. But the continued pressure of combined effort, provided that the ideal is really present in the minds of the mass of teachers, can do much, and effects in the end surprising modification. Gradually the requisite books get written, still more gradually the examinations are reformed so as to give weight to the less technical aspects of the subject, and then all recent experience has shown that the majority of teachers are only too ready to welcome any practicable means of rescuing the subject from the reproach of being a mechanical discipline.

REVIEWS.

On the Foundation and Technic of Arithmetic. By GEORGE BRUCE HALSTED. Pp. iv + 133. 1 dollar. 1912. (Chicago: The Open Court Company.)

The leading characteristic of this book is enthusiasm. On the first page, we read: "In arithmetic a child may taste the joy of the genius, the joy of creative activity." And the enthusiasm over the "creation" of numbers pervades the whole book (cf. pp. 3, 4, 69), and reminds one (as do other passages on pp. 10, 111) of Dedekind. Certain modern advances in the logic of mathematics are referred to: "logicization" is spoken of (p. 1), the cardinal number of a set *s* "is the class of all sets similar to *s*" (p. 6), a one-to-one correspondence is treated as fundamental (p. 10), and the *infinite* is defined (p 82). Yet Dr. Halsted's interest is mainly psychological: indeed, he repeats (p. 86) Poincaré's groundless sneer against the syllogism. In psychology—in his treatment, with the aid of history, of the teaching of the rules and technique of arithmetic and of the merits of its symbolism and numerical systems—Dr. Halsted is very stimulating. "There is a logic of it [the theory of arithmetic], yet the child need not necessarily know, had perhaps better not know, that logic. The teacher should know, the child practise," he writes on p. 23. I think the teacher will learn more psychology than logic from this book, and probably psychology will be more useful to him. But the refreshing and almost boyish enthusiasm with which this book is written will be more useful still.

PHILIP E. B. JOURDAIN.

An Introduction to Algebraical Geometry. By A. CLEMENT-JONES, M.A., Ph.D., Senior Mathematical Master at the Bradford Grammar School. Pp. 548. 1912. (Clarendon Press.)

Analytical Geometry has long been one of the most important of the subjects taught to mathematical students, both as a subject of study in itself and as an introduction to the methods of dealing with such branches as Higher Geometry, Theory of Functions and Analytical Physics. The publication of a careful treatise on the subject is therefore of importance to teachers.

In the present work Dr. Clement-Jones has embodied the results of his twelve years' experience as a teacher of scholarship candidates. The book consists of 548 pages, comprising bookwork and examples. It covers the ground up to and including Honour Moderations at Oxford. It is divided into twelve chapters. The first four chapters deal with the Point, the Equation of the First Degree, Change of Axes, Imaginaries and Points at Infinity. The next four chapters treat of the general Equation of the Second Degree and in particular of the Circle, Parabola, Ellipse and Hyperbola. The last four chapters discuss such analytical instruments as Polar Coordinates, Homogeneous Coordinates, etc.

The merits of the book are many. It recognizes the importance of an early introduction to accurate conceptions of the Line at Infinity and Imaginary Elements in relation to geometry. The discussions of these subjects in Chapter IV. and of the Circular Points at Infinity on page 206 are amongst the best parts of the book. The introduction to tangential coordinates and equations on page 392 and onwards is especially good, the analogy between the point and line properties being emphasized throughout and tabulated in contrast. The properties of the Parabola are investigated mainly by means of parameters, and the student will gain from reading this chapter an excellent introduction to this method,—so potent in all higher geometry. Methods are also given on pages 481-484 of writing down instantaneously the point and line equations of circles in areal coordinates. The theoretical processes are copiously illustrated by worked-out examples.

On the other hand, the treatment is too detailed throughout. Much of the bookwork could have been treated as examples and much omitted. Most students would not be able to see the wood for the trees. For example, Oblique Axes, Change of Axes, and Metrical Invariants are dealt with at great length in Chapter III. In solving questions in practice, these heavy analytical methods can usually be avoided, and the absence of complicated calculation allows the operator to see the geometrical aspect of the problem more clearly. Furthermore, the principle of "vocational training" applies in mathematics as in other branches of pedagogy, and the study of such intricate analysis should be postponed until the geometrical properties to be investigated require the use of it. Otherwise, the student gets discouraged because he cannot see what he is "getting at." Again,

Dr. Jones's experience leads him to introduce "abridged notation" very early on page 52, and he gives many examples of its use. Probably the experience of most teachers would lead them to defer such methods till the end of the book. One has only to remember one's own difficulties in reading Salmon on this subject. "Umbral expressions" are very difficult for the beginner to grasp and manipulate. The teaching of the beginner must be very concrete. Also, the introduction to the Circle in Chapter V. will be found unnecessarily difficult. The usual *direct method* of obtaining its equation, given the centre and radius, is very simple and much to be preferred.

It would have been better if, all through the book, the geometry had been kept more in prominence and the calculations kept more in check. The introduction to the General Conic is very analytical and is difficult. A frank recognition of the well-known elementary properties of the Central Conics would simplify their treatment a good deal. This is contrary to Dr. Jones' explicitly expressed views in the preface, where he says: "The author has tried to avoid obtaining analytical results by quoting geometrical results with which the reader may be acquainted in Geometrical Conics. This process makes some pupils lose confidence in Analytical Geometry; others welcome it as a dodge enabling them to avoid a real understanding of the principles of the subject; in either case the result is bad. All properties of the conic are developed by analytical processes following on definitions." Now the successful investigation of geometrical properties lies neither in an excessive restriction to the methods of pure geometry nor in a wholesale adherence to the processes of algebraic analysis, but in a judicious combination of both. Most teachers are familiar with the three closely written foolscap sheets of x's and y's and formulae getting more and more out-of-hand, shown up with a despairing shake of the head when the pupil's courage at last gives out, a frequent phenomenon to which the practical application of the above quotation inevitably leads. Lastly, to give another example, the general formulae for the lengths of the axes, the eccentricity of the conic, the radius of curvature, etc., in areal coordinates are excessively complicated, and might advantageously have been omitted.

If much of the detailed work had been left out, many teachers would have welcomed an elementary treatment of such subjects as the geometrical interpretation of the invariants Δ, Θ, etc., the F and Φ conics and the simpler applications of their theory. A field of geometry, at once elegant and simple, is thus opened up to the student, and the profit is greater than that afforded by interminable manipulative analysis.

The general impression left by a careful perusal of the book is that, while it will prove an invaluable book of reference for the teacher, it will be found less suitable for the learner owing to its being overloaded with manipulative detail. Also, it would have been better if the bookwork had been kept more distinct from the worked-out illustrative examples in order to facilitate rapid revision when necessary.

On the other hand, the work has been executed with great care and thought, and there is an excellent collection of examples. WILLIAM P. MILNE.

THE LIBRARY.

THE Librarian acknowledges with thanks Saunderson's *Method of Fluxions*, 1756, presented by Mr. Joslin. Also *Sectiones Conicae*, by de la Hire, 1685, presented by Rev. J. J. Milne.

The Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue has been issued to members containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

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